

ESSAYS ON SECULAR STAGNATION

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Introduction

Interest rates and inflation have been persistently low in relevant world economies. When the zero lower bound for the nominal interest rate prevents the real interest rate to reach an eventually negative full employment level - the *Natural Rate of Interest* - an economy may enter into a stable recession, according to recent work on *Secular Stagnation* [34] and *Liquidity traps* [31]. Moreover, when the zero lower bound for nominal interest rates binds, short term monetary policy may lose effectiveness.

Understanding the factors that may drag down the *Natural Rate of Interest* in a permanent way is a fundamental step in order to design the most adequate policies to overcome or to avoid the undesirable consequences of a prolonged economic slump. Some of those factors have been explored in recent literature, where we can namely highlight diminishing borrowing limits during longstanding deleveraging shocks in Eggertsson and Mehrotra [21] recent work on *Secular Stagnation*, or demographic changes inspected for example by Carvalho et al. [12]. Inspecting the mechanisms dragging down the *natural rate of interest* in a longstanding way may require alternative modeling options to the standard single agent RBC and New Keynesian frameworks, where negative real interest rate levels are generally temporary by construction. Moreover, the policy prescriptions to counteract a temporary recession may not necessarily be applicable when dealing with a lasting economic slump, where the impact of a policy must be permanent. For example, when a

higher level of inflation is not a desirable policy outcome to allow a first best solution consistent with a negative *natural rate of interest*, Correia et al. [16] propose an emulation of the role of higher consumer price inflation through a temporary stable increase of consumption taxes in a single agent NK model. The transitory nature of that prescription might not be useful if the *natural rate of interest* is permanently low; in contrast with alternative policies that may permanently level-up the *natural rate of interest* to a level where monetary policy becomes effective again.

Our research aims at complementing recent work on counteracting persistent recessions, as well as analytically inspecting mechanisms with persistent impact on the natural rate of interest. In particular, our contribution in Chapter 1 is to inspect the role of distortionary taxes in avoiding a permanent slump, complementing recent work on *Secular Stagnation*, and on the role of distortionary taxation in temporary recessions. Our contribution in Chapters 2 and 3 is to inspect analytically the impact of changing age milestones and inequality on the *natural rate of interest*, by deriving explicit algebraic relations between real interest rate changes and changes on those factors.

In the first Chapter we formalize the role of distortionary taxation in avoiding a stable recession characterized by the *Secular Stagnation* framework proposed by Eggertsson and Mehrotra [21], based on a three generations OLG model with borrowing constraints. We compare our results with the ones obtained by Correia et al. [16] that propose a solution based on the same set of distortionary taxes in a standard single agent New Keynesian model without borrowing constraints, to counteract a liquidity trap that is by construction temporary. We find reversed results. Our mechanism is based on a wealth re-distributive policy using distortionary taxes to increase the natural rate of interest so that it becomes achievable given the monetary policy targets, by increasing labor taxes on the middle age employed and redistributing the tax proceeds to the population in general by reducing consump-

tion tax. Instead, Correia et al. [16] emulate inflation in consumer prices using an increasing path of consumption taxes, so that the intertemporal condition allows an achievable negative natural rate of interest, and the liquidity trap is neutralized. We use the same fiscal toolbox with different approaches. We increase the natural rate of interest to a level consistent with monetary policy effectiveness, instead of temporarily allowing a first best solution compatible with a negative natural rate of interest.

As an alternative to the standard fiscal policy prescriptions to counteract an economic downturn, based on Keynesian increases of public expenditures, more public debt and tax cuts to stimulate demand¹, and complementing the papers referenced above, the main purpose of our analysis is to show how fiscal policy based on distortionary taxation can be effective in avoiding persistent recessions, even when increasing public expenditures and debt are not policy options available.

In the second chapter we formalize the relation between real interest rates and relevant age milestones of an agent's life, using an overlapping multi-generations model where one generation correspond to one year. Although the impact of age structure in relevant World economies has been a recurrent topic covered in recent literature, in particular to explain the persistent decline of interest rates and economic stagnation, there has not yet been an attempt, to the best of our knowledge, to formally derive the analytic relations of real interest rates to each age milestone. The purpose of this chapter is to fill out this gap, by deriving tractable algebraic real interest rate elasticity expressions with respect to each age parameter, in order to formalize the mechanisms by which real interest rate changes occur. This allows, for example, to algebraically derive precisely by how much the natural rate of interest may be permanently dragged down eventually to negative levels, by any combination of increasing life expectancy, postponing first child birth, lowering the

¹Besides other non-fiscal approaches, namely the one proposed by Eggertsson and Woodford [23] where the central bank commits to keep interest rates at a lower level even after a recession resulting from a liquidity trap is over.

retirement age, increasing adulthood age, or reducing the age of first job. Moreover, we can use our framework to quantify explicitly the impact on the reduction of real interest rate level in recent years, of the evolution of population age structure. The main underlying mechanism relating age milestones and real interest rates in our model relies on the relative place and duration of labor income with respect to life expectancy when agents smooth consumption. For example, a longer retirement period, resulting from a reduction of the retirement age or an increase of life expectancy, makes households save more, thus expanding the supply of loans that drags down the real interest rate that ensures equilibrium in the loan market. We also inspect how exogenous parameters to our model, namely the elasticity of intertemporal substitution, productivity growth, income growth path of households, and population growth, may amplify or mitigate the impact of age milestones changes on the natural rate of interest. In addition we show how inter-generation transfers are affected by age milestone changes. For example, why and how increasing life expectancy may reduce endogenous bequest levels, decrease the propensity to help children, or increase the willingness to help parents. Laterally to our main contribution in this chapter, the analytic formulation of interest rate changes with respect to age milestones, we also developed a tractable algebraic toolbox to solve overlapping multi-generations optimization problems.

Finally in Chapter 3, we formalize the relation between increasing income inequality and low real interest rates using an overlapping generations model with borrowing constraints and a bequest motive. Our main contribution in this chapter is to present in a single simple framework an explicit formalism linking low real interest rates with increasing income inequality, by gathering and building on some relevant topics in recent literature, namely increasing income inequality[33], bequests[4][7], the question of whether higher-lifetime income levels lead to higher marginal propensity to save[19] and lower marginal propensity to borrow[1][32], and *secular stagnation*[21][34]. The underlying mechanism in our model relating real interest rates

and inequality is based on empirical evidence in recent literature that households' marginal borrowing and saving rates are respectively negative and positive functions of income, so that the net effect on aggregate borrowing and savings of a permanent increase of income inequality is respectively a net contraction and a net expansion, that may lead to a persistent reduction of the *natural rate of interest*. In particular, the borrowing mechanism in our model is based on the concavity of the marginal propensity to borrow, and on binding borrowing constraints, both consistent with empirical observations in recent literature[1][32]. In addition, the saving mechanism is illustrated with an endogenous propensity of households to leave a bequest when they expect the next generation to be poorer. So that wealthier households are more generous, making their marginal savings rate higher than the poor that in general leave much lower or no bequests at all, as also observed by Hendricks [28].

Chapter 1

Distortionary Taxation in Stable Recessions

ABSTRACT

Increasing public spending, generating budget deficits, or raising the level of public debt, may not be options available for an economy trying to avoid a recession. Complementing recent work on *Secular Stagnation* and on fiscal policy during liquidity traps, we use an overlapping generations New Keynesian model with borrowing constraints to explore how *distortionary taxes* can be used to circumvent a persistent economic slump, by raising the full-employment equilibrium real interest rate so that it becomes achievable when the nominal interest rate zero lower bound is binding. We propose a wealth redistributive fiscal policy, by taxing labor income and reducing consumption or capital income taxes, leading to a contraction of savings that triggers an increase of the *natural rate of interest*¹. We compare our results with an alternative approach in recent literature based on emulating inflation in consumer prices using an increasing path of consumption taxes, so that the intertemporal condition allows an achievable negative *natural rate of interest* to neutralize a liquidity trap.

¹Full-employment equilibrium real interest rate.

1.1 Introduction

During a liquidity trap, when inflation is low and the zero lower bound for nominal interest rates prevents the real interest rate to reach its full employment level, short term monetary policy may lose effectiveness.

To circumvent a liquidity trap, standard fiscal policy prescriptions in the literature have been based on Keynesian increases of public expenditures and tax cuts to stimulate demand², and thus increase inflation, allowing for a reduction of the real interest rate towards the *natural rate of interest*³. More recently, Correia et al. [16] proposed an alternative approach based on the use of distortionary taxes, with *"no need to use inefficient policies such as wasteful public spending or future commitments to low interest rates"*. To allow a first best solution compatible with a negative *natural rate of interest* when the zero lower bound for nominal interest rates is binding, their recipe consists of emulating the role of higher consumer price inflation, by a stable increase in consumption taxes. They use a standard single agent New Keynesian model, where a slump is by construction temporary.

In this paper we also inspect how fiscal policy based on distortionary taxation may circumvent a liquidity trap, but in the context of a permanent slump. Our approach is based on the recent *Secular Stagnation* literature, in particular on the work of Eggertsson and Mehrotra [21], to which we add a module of distortionary taxes. We use their proposed three generations OLG model, where the natural rate of interest may be persistently negative. In contrast with the *inflation emulation* fiscal policy type proposed by Correia et al. [16], that increases gross consumption prices through an increase of consumption taxes to allow a model first-best solution, the distortionary taxation prescription to avoid a liquidity trap in a permanent slump

²Besides other non-fiscal approaches, namely the one proposed by Eggertsson and Woodford [23] where the central bank commits to keep interest rates at a lower level even after a recession resulting from a liquidity trap is over.

³The natural rate of interest has been defined in the literature as the equilibrium full-employment real interest rate.

using the model proposed by Eggertsson and Mehrotra [21] is based on a wealth redistribution mechanism from an endowed middle aged generation, to young credit constrained households, and to the older generation living at the same time. An increase of income taxes on the middle-age would reduce their net available funds to consume and to save, consequently⁴ triggering an increase of the *natural rate of interest* to a higher stable level. If the proceeds from this tax increase are used to reduce the consumption tax, then the borrowing constrained younger households, as well as the old, would be able to consume more.

Interestingly, although this paper and the one of Correia et al. [16] both use distortionary taxation to allow first best, full-employment solutions to circumvent the potential damages of a liquidity trap, the prescriptions are reversed. Correia et al. [16] neutralize the effects of the zero lower bound to achieve negative real interest rate levels, by inducing inflation in consumer prices, the ones that matter for intertemporal decisions, with an increasing path of consumption taxes over time, simultaneously reducing labor taxes such that producer price inflation is kept at zero. They use a single agent standard New Keynesian model, with no borrowing constraints, where a slump is by construction transitory. In contrast, in our model there are three types of households, where the younger generation is credit constrained. We use distortionary taxation to increase the natural rate of interest to an achievable level, instead of neutralizing obstacles to achieve a negative level. Instead of inducing inflation in consumer prices to achieve negative real interest rates, our prescription is wealth redistributive in order to increase the natural rate of interest. The fiscal policy we propose could be implemented with lump-sum taxes and transfers, from middle age to young and old agents. Lump-sum taxation is then effective in our heterogeneous agents world with borrowing constraints, but not in a standard single agent New Keynesian framework. Last but not least, we propose an

⁴assuming a standard context where the derivative of excess savings with respect to the real interest rate is positive. Excess savings being defined as the difference between loan supply and demand.

increase of labor taxes to contract savings and increase the natural rate of interest, allowing also for a reduction of consumption taxes to the relative benefit of younger and older agents; in contrast with a reversed prescription by Correia et al. [16] based on increasing consumption taxes and reducing labor taxation.

This chapter is then organized as follows. In the next section we inspect how distortionary taxes may affect the natural rate of interest, by using a general version of the model introduced by Eggertsson and Mehrotra [21] with *Constant Relative Risk Aversion* household preferences. We show that *emulating inflation* with a stable increasing path of consumption taxes would not be able to increase the natural rate of interest back to positive ground on a stable basis, although it could be effective in the short term, but only when the elasticity of intertemporal substitution is greater than one. In addition, we show that the fiscal policy at the core of this chapter, based on taxing income of middle age households, is effective in the short and long run, ensuring an increase of the natural rate of interest to a stable level irrespectively of the elasticity of inter-temporal substitution⁵. Without loss of generality, we assume EIS ⁶ is equal to one through the rest of the chapter because it allows closed-form expressions in particular for the equilibrium real interest rate. We then use log-utility of consumption preferences to inspect the role of each fiscal instruments in avoiding a liquidity trap by sustaining the natural rate of interest at positive levels. As an example, we present closed-form solutions for the changes required for each tax in order to offset the impact of deleveraging shocks on the natural rate of interest.

In the third section we introduce nominal prices and endogenous rigid wages to model a *Secular Stagnation* equilibrium as in Eggertsson and Mehrotra [21]. Capital and a tax on capital income are introduced in the last section. We inspect the role of distortionary taxation in avoiding a stable recession, driven in this model by a

⁵Same assumption as before about the slope of excess savings with respect to real interest rate.

⁶ EIS is the acronym of *Elasticity of Intertemporal Substitution*

deflation mechanism that prevents a timely reduction of nominal rigid wages, that triggers an increase of real wages to a stable level away from the full-employment flexible wage, thus dragging the economy into a persistent recession. We remind[21] that monetary policy can ensure the existence of a full-employment equilibrium, but cannot by itself avoid its coexistence with a *Secular Stagnation* one, to which the economy can transit, namely during a deleveraging shock, if fiscal policy is not used in a sufficiently assertive way. Adequate fiscal policy measures based on distortionary taxation promoting the expansion of aggregate demand, possibly combined with the contraction of aggregate supply, may effectively prevent the economy to fall into a slump, or ensure that it transits out of one.

Reducing consumption and capital income taxes expand aggregate demand, which combined with an increase of labor taxes, may raise the natural rate of interest, eventually to achievable levels altogether neutralizing a stable recession. Furthermore, and reminding the *Paradox of Toil*[20], a demand expansion can be combined with a supply contraction that in the short term can boost inflation and force the transition from a stable recession with deflation, to a positive inflation full-employment equilibrium. This can be achieved by increasing the labor tax on firms that would lead firms to reduce nominal wages in order to sustain profitability. The nominal wage rigidity prevents the nominal wage full adjustment triggering a further increase of real wages which contracts supply even further, thus creating a positive pressure on inflation that may bring back the economy to a higher employment level. The results presented in this chapter are related to a closed economy, although the combination alternatives of distortionary taxes to avoid a permanent slump, for example by using a reduction of the tax on capital income instead of the consumption tax, suggest that this framework could be applied to other contexts, namely to open economy frameworks.

1.2 *Secular Stagnation* model with distortionary taxes

In this section we briefly describe and derive a *Secular Stagnation* overlapping three generations model with real prices proposed by Eggertsson and Mehrotra [21], to which we add a module of distortionary taxes. The aim is to find a general expression for the impact of tax changes on the natural rate of interest, so that it can be kept at a sufficiently high achievable level, for example during a deleveraging shock. We use the *Implicit Function Theorem* to derive a general expression for the partial derivatives of the natural rate of interest with respect to policy instruments, to explore the effectiveness of fiscal policy options in avoiding a liquidity trap, by manipulating the natural rate of interest level. In particular we show that *inflation emulation* fiscal approach based on an increasing path of consumption taxes proposed by Correia et al. [16] is not effective in counteracting a persistent recession in the context of our model.

1.2.1 Model with labor and consumption taxes

We use a 3 periods overlapping generations model[21] that allows for steady state equilibria with persistent negative real interest rates, when population growth is low enough or when borrowing constraints of the younger generation increase. Households go through three stages of life: young, middle aged and old. The young generation borrows B_t^y from the middle-aged; the middle-aged save $-B_t^m$ by lending to the young and to the government, and pay back their loans to the previous generation, the old. The old receive back with interest what they have lent to the young and the Government when middle aged. Consumption of the young C_t^y , middle age C_t^m , and old C_t^o is taxed by a consumption tax τ_t^c . It is assumed that income is only earned by the middle aged through firm profits Z_t , and labor $W_t L_t$

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which is taxed with τ_t^l . Borrowing is constrained by a binding debt limit D_t faced by the young, exogenously determined. The Government budget is balanced. The household objective function is given by:

$$\max_{C_t^y, C_{t+1}^m, C_{t+2}^o} E_t \{U(C_t^y) + \beta U(C_{t+1}^m) + \beta^2 U(C_{t+2}^o)\} \quad (1.1)$$

$$\text{s.t. } (1 + \tau_t^c)C_t^y = B_t^y \quad (1.2)$$

$$(1 + \tau_{t+1}^c)C_{t+1}^m = Z_{t+1} + W_{t+1}L_{t+1}(1 - \tau_{t+1}^w) - (1 + r_t)B_t^y + B_{t+1}^m \quad (1.3)$$

$$(1 + \tau_{t+2}^c)C_{t+2}^o = -(1 + r_{t+1})B_{t+1}^m \quad (1.4)$$

$$(1 + r_t)B_t^y \leq D_t, \text{ an exogenous borrowing limit.} \quad (1.5)$$

Where $U(C)$ is a constant elasticity of inter-temporal substitution utility function expressed by $U(c) = \frac{c^{1-\sigma}}{1-\sigma}$. We assume that the borrowing constraint of the young generation is binding, or $(1 + r_t)B_t^y = D_t$. In this credit constrained environment, although the amount borrowed by a young agent does not directly depend on any fiscal instrument but just on the real interest rate and the borrowing constraint, $B_t^y = \frac{D_t}{1+r_t}$, the consumption of the young will inversely directly depend on the level of consumption taxation, given by:

$$(1 + \tau_t^c)C_t^y = B_t^y = \frac{D_t}{1 + r_t} \quad (1.6)$$

So unless an increase of consumption taxes were followed by a sufficient reduction of equilibrium real interest rates, they would result in a consumption contraction of younger agents. The same could be said about the old, but in this case with no relation with current equilibrium interest rate, since their consumption depends on savings from previous period, net of consumption taxes:

$$(1 + \tau_t^c)C_t^o = -(1 + r_{t-1})B_{t-1}^m \quad (1.7)$$

Consequently an increase of consumption taxation, as proposed by Correia et al. [16], in this environment would lead to a contraction of consumption of the younger and older generations, with an expansion of middle age consumption, assuming stable government spending and full-employment. A negative welfare effect if we assume that middle age are wealthier than the young and old.

As the consumption of the old is determined by previous period equilibrium, unconstrained middle age consumption is determined by the model's Euler equation given by:

$$1 + r_t = \frac{1}{\beta} \mathbb{E}_t \frac{U_c(C_t^m)}{U_c(C_{t+1}^o)} \frac{1 + \tau_{t+1}^c}{1 + \tau_t^c} \quad (1.8)$$

This Euler equation is similar to the one derived by Correia et al. [16], and would suggest that an increase of consumption taxation could have an increasing effect on the equilibrium real interest rate as well. Next we show why this is not necessarily the case in our borrowing constrained economy where persistent recessions are possible.

1.2.2 Increasing the natural rate of interest

The model is completely determined with the budget constraints, the previous Euler equation, and the loan market equilibrium equation given by:

$$B_t^y = -B_t^m \quad (1.9)$$

where $B_t^y = L_t^d$ is the demand for loans by the young generation, which in equilibrium must be equal to the supply of loans from the middle aged $-B_t^m = L_t^s$. Let *Excess Savings* be a continuous and differential function for $r \in]-1, +\infty[$ defined as $S_t = L_t^s - L_t^d$, such that at loan market equilibrium $S_t = 0$. Using the the *Implicit Function Theorem* we can explicit the partial derivative of the natural rate of interest with

respect to any tax parameter:

$$L_t^d = L_t^s \Leftrightarrow S_t = 0 \Rightarrow \frac{\partial r_t}{\partial \tau_t} = -\frac{\frac{\partial S_t}{\partial \tau_t}}{\frac{\partial S_t}{\partial r_t}} \quad (1.10)$$

We now inspect each component of the previous expression.

i) Loan Demand

Total borrowing in period t is equal to the total demand for loans, and is given by total borrowing of young households and the government, from the middle age:

$$B_t = N_t B_t^y + N_{t-1} B_t^g \quad (1.11)$$

where B_t^g is government borrowing per middle age household, N_t is the size of the young generation at time t , and N_{t-1} is the size of middle age generation also at time t . Then the demand for loans per middle age agent is given by:

$$L_t^d = \frac{B_t}{N_{t-1}} = (1 + g_t) B_t^y + B_t^g = \frac{1 + g_t}{1 + r_t} D_t + B_t^g \quad (1.12)$$

where $1 + g_t = N_t/N_{t-1}$, and g_t is population growth at time t . In that case a reduction of the borrowing limit of the young would cause a contraction of loan demand, that could be counterbalanced by an increase of public debt. Assuming that government borrowing is exogenously determined, loan demand would not depend on any tax instrument in this credit constrained environment, implying that $\frac{\partial S_t}{\partial \tau_t} = \frac{\partial L_t^s}{\partial \tau_t}$. Moreover, loan demand is a negative function of the real interest rate, with a strictly negative partial derivative given by:

$$\frac{\partial L_t^d}{\partial r_t} = -\frac{1 + g_t}{(1 + r_t)^2} D_t = -\frac{L_t^d (1 - B_t^g/L_t^d)}{1 + r_t} \quad (1.13)$$

Because we assumed that government borrowing B_t^g is an exogenous parameter of this economy, we can already simplify the general expression for the partial derivative

of the natural rate of interest with respect to tax instruments to conclude that their relative sign depends on the impact of the tax instrument on loan supply.

$$\frac{\partial r_t}{\partial \tau_t} = -\frac{\frac{\partial S_t}{\partial \tau_t}}{\frac{\partial S_t}{\partial r_t}} = -\frac{\frac{\partial L_t^s}{\partial \tau_t}}{\frac{\partial S_t}{\partial r_t}} \quad (1.14)$$

As we show later, we can realistically assume that excess savings in the denominator is an increasing function of the real interest rate. In that case any tax change having a contraction effect on loan supply, would raise the *natural rate of interest*. We then derive loan supply.

ii) Loan Supply

Firms in the model are in perfect competition and hire labor to maximize profits on a period by period basis. For now there are no distortionary taxes on firms. The firm problem is given by:

$$Z_t = \max_{L_t} Y_t - W_t L_t \text{ s.t. } Y_t = F(L_t) \text{ strictly concave} \quad (1.15)$$

The solutions for W_t and Z_t are given by:

$$W_t = F_L(L_t) \text{ and } Z_t = F(L_t) - L_t F_L(L_t) \quad (1.16)$$

We then get a present value expression for consumption of middle age and old by combining constraints (1.3) and (1.4) with the borrowing constraint (1.5), and replacing the solutions for W_t and Z_t :

$$(1 + \tau_t^c)C_t^m + \frac{(1 + \tau_{t+1}^c)C_{t+1}^o}{1 + r_t} = F(L_t) - \tau_t^w F_L(L_t)L_t - D_{t-1} \quad (1.17)$$

Loans supply is derived by replacing expressions for middle age and old consumption

given by (1.8) and (1.7) in previous equation:

$$L_t^s = -B_t^m = \frac{[F(L_t) - \tau^l F_L(L_t)L_t] - D_{t-1}}{1 + \frac{1}{\beta^{\frac{1}{\sigma}}} \left[(1 + r_t) \frac{1 + \tau_t^c}{1 + \tau_{t+1}^c} \right]^{1 - \frac{1}{\sigma}}} \quad (1.18)$$

We see directly from this expression that an increase of labor taxation on middle age is enough to trigger a contraction of loan supply, independently of the elasticity of inter-temporal substitution $EIS \equiv \frac{1}{\sigma}$. By realistically assuming that excess savings increases with the real interest rate ⁷ a permanent increase of labor taxation on the middle age would then trigger a permanent increase of the natural rate of interest. Consumption taxes would have to be reduced in order to keep government budget balanced, in current and following periods. Furthermore, if we assumed that consumption taxes would remain constant in the present and future, then loan supply would not depend on consumption taxes, but on labor taxes alone which would become the only fiscal instrument directly determining loan supply:

$$L_t^s = \frac{[F(L_t) - \tau^l F_L(L_t)L_t] - D_{t-1}}{1 + \frac{1}{\beta^{\frac{1}{\sigma}}} (1 + r_t)^{1 - \frac{1}{\sigma}}} \quad (1.19)$$

An *inflation emulation* policy with a stable increase of consumption taxes can here be given by a commitment to increase consumption taxation in the next period. The impact on loan supply depends on EIS , as can also be observed directly from expression (1.18). If $EIS \equiv \frac{1}{\sigma} < 1$, then committing to increase consumption taxation in the next period expands loan supply in the current period, and reduces the natural rate of interest. With a low elasticity of inter-temporal substitution, the income effect related to an expected increase of consumption taxation in the next period would lead to an increase of savings in the current period to sustain future

⁷As we show later, a sufficient condition for a positive slope of excess savings S_t with respect to r_t is that the ratio of government borrowing over total borrowing in the economy $\frac{B_t^g}{L_t^g}$ does not exceed the elasticity of inter-temporal substitution $\frac{1}{\sigma}$. This is a reasonable assumption in developed economies, given the standard estimations of EIS usually above 0.5[27]. If EIS is close to 1, or for low level of government debt B^g this becomes a trivial assumption.

consumption. This effect would here prevail over a substitution effect resulting from the fact that consumption in the future would become more *costly* than current consumption. The opposite is true if $EIS > 1$, which may not be the most usual case taking the results of Havranek et al. [27]. Then a substitution effect would prevail with a commitment to increase consumption taxes in the future, leading to a contraction of current savings, and to a consequent increase of the natural rate of interest. But in this case, for the budget to balance during the next period labor taxes would have to decrease, creating a negative pressure on the natural rate of interest as seen before, that would need a further commitment to increase consumption taxation in the future, and so on. This fiscal policy, besides being effective only for $EIS > 1$, would force an unimplementable recurrent policy commitment from the government, with no opportunity for reversion in the context of the current model were a recession may be persistent.

iii) *Slope of Excess Savings with respect to the real interest rate: $\frac{\partial S_t}{\partial r_t} > 0$?*

We first derive an expression for the derivative of loan supply with respect to real interest rate:

$$\frac{\partial L_t^s}{\partial r_t} = \left(\frac{1}{\sigma} - 1 \right) \frac{L_t^s}{1 + r_t} \frac{1}{1 + \beta^{\frac{1}{\sigma}} \left(\frac{1}{1+r_t} \frac{1+\tau_{t+1}^c}{1+\tau_t^c} \right)^{\frac{\sigma-1}{\sigma}}} \quad (1.20)$$

Loan supply is positive sloped for $EIS > 1$. Since $\frac{\partial S_t^s}{\partial r_t} = \frac{\partial L_t^s}{\partial r_t} - \frac{\partial L_t^d}{\partial r_t}$, and $\frac{\partial L_t^d}{\partial r_t} < 0$, if $EIS > 1$ then $\frac{\partial S_t}{\partial r_t}$ is positive. If $EIS < 1$ then loan supply is negative sloped with respect to r . A sufficient condition for $\frac{\partial S_t}{\partial r_t} > 0$ is $\frac{B_t^g}{L_t^d} < EIS$ ⁸.

We assume that inequality $\frac{B_t^g}{L_t^d} < \frac{1}{\sigma}$ is valid through the rest of the paper. This implies that the sign of the partial derivative of the natural rate of interest with respect to any tax instrument is opposite to the sign of the partial derivative of loans supply

$$\begin{aligned} \frac{\partial S_t}{\partial r_t} > 0 &\Leftrightarrow -L_t^d > -L_t^s \Leftrightarrow \frac{L_t^d(1-B_t^g/L_t^d)}{1+r_t} > \left(1 - \frac{1}{\sigma}\right) \frac{L_t^s}{1+r_t} \frac{1}{1+\beta^{\frac{1}{\sigma}} \left(\frac{1}{1+r_t} \frac{1+\tau_{t+1}^c}{1+\tau_t^c} \right)^{\frac{\sigma-1}{\sigma}}} \Leftrightarrow 1 - B_t^g/L_t^d > \\ &1 - \frac{1}{\sigma} \Leftrightarrow \frac{B_t^g}{L_t^d} < \frac{1}{\sigma}. \end{aligned}$$

with respect to that same instrument. If we further discard from our analysis policy measures supported by increasing (or decreasing) paths of consumption taxes, since we checked them previously, we can assume without loss of generality, and for the purpose of algebraic tractability, that $EIS = 1$, and $U(C) = \text{Log}(C)$.

1.2.3 Offsetting a deleveraging shock

In this section we explore each fiscal instrument in terms of its role in offsetting any factor dragging down the natural rate of interest. We start with labor taxation on households.

i) Labor tax on households: τ^l

For $\sigma = 1$, a closed-form expression for the natural rate of interest can be derived directly from loan market equilibrium given by equation (1.12):

$$1 + r_t^n = \frac{(1 + g_t)D_t}{\frac{\beta}{1+\beta} \left[(1 - \alpha\tau_t^l)Y_t^f - D_{t-1} \right] - B_t^g} \quad (1.21)$$

Where we assume that output is now expressed by $Y_t = L_t^\alpha$, and in particular, full-employment output is given by $Y_t^f = \bar{L}^\alpha$ when the equilibrium real interest rate is equal to the natural rate of interest r_t^n . We can observe directly from expression (1.21) that the natural rate of interest decreases with the contraction of aggregate borrowing, either because the Government needs to reduce public borrowing B_t^g , or because of higher credit constraints imposed on young households through lower borrowing limits D_t . If the government is prevented to increase public debt, then a sufficient increase of labor taxation τ_t^l could offset the impact of a deleveraging shock, by triggering a contraction of aggregate savings to prevent the natural rate of interest from falling to non-achievable (negative) levels. In particular, to counteract the impact of a deleveraging shock, from a reduction of public borrowing, on the

natural rate of interest, the required change of labor tax would be:

$$\partial\tau_t^l = -\frac{1}{\alpha} \left(\frac{1+\beta}{\beta} \right) \frac{\partial B_t^g}{Y_t} \quad (1.22)$$

and to offset the impact on the natural rate of interest of a deleveraging shock, due to a lower borrowing limit of young households, the required change of labor tax would be⁹:

$$\partial\tau^l = -\frac{1}{\alpha} \left[1 + \left(\frac{1+\beta}{\beta} \right) \frac{1+g}{1+r^n} \right] \frac{\partial D}{Y^f} \quad (1.23)$$

ii) *Consumption tax: τ^c*

With no capital in the model, aggregate demand is the sum of aggregate consumption C_t and government spending G_t . If the government is prevented to spend more than an exogenous upper limit G , then consumption C_t cannot be lower than $C^f = Y^f - G$ in order to ensure full-employment. By assuming that public spending is exogenously determined we next explore how to sustain consumption at its full-employment level, combining distortionary taxes so that the natural rate of interest is also sustained at viable levels. Total consumption in period t is given by:

$$C_t^{Total} = N_{t-1}C_t = N_t C_t^y + N_{t-1}C_t^m + N_{t-2}C_t^o$$

Combining the previous expressions with young and old budget constraints (1.6),(1.7), the euler equation (1.8), and loan market equilibrium (1.12), consumption is also expressed by:

$$C_t = \frac{1}{1+\tau_t^c} \left\{ \left[\left(\frac{1+\beta}{\beta} \right) \frac{1+g_t}{1+r_t} D_t + D_{t-1} \right] + \left[\frac{B_t^g}{\beta} + \frac{1+r_{t-1}}{1+g_{t-1}} B_{t-1}^g \right] \right\} \quad (1.24)$$

Consumption is directly determined by the borrowing constraints of the government

⁹This is a steady state expression.

and the young generation. Because the borrowing constraints are binding, consumption inversely depends on consumption tax. Then, with a contraction of the borrowing limit of younger households, the effective fiscal instrument that sustains the consumption level from falling is τ^c , if the real interest rate is to be sustained too. Further below, and considering the government budget constraint, we will show that the change in labor income tax required to prevent the real interest rate from falling after a deleveraging shock is consistent with the required reduction of the consumption tax that prevents consumption level from falling, given the same deleveraging shock.

The consumption tax changes required to sustain the same level of consumption, given a reduction of households borrowing limit, or of government debt, assuming the real interest rate is sustained too, are respectively given by the following expressions in steady state:

$$\partial\tau^c = \left[\left(\frac{1+\beta}{\beta} \right) \frac{1+g}{1+r} + 1 \right] \frac{\partial D}{Y-G} \quad (1.25)$$

$$\partial\tau^c = \left[\frac{1}{\beta} + \frac{1+r_s}{1+g} \right] \frac{\partial B^g}{Y-G} \quad (1.26)$$

Directly from expressions above, to counteract a deleveraging shock, the fiscal adjustment would need to be stronger for governments that spend more.

iii) Government Budget constraint

Until now we used two approaches to offset the impact of a deleveraging shock respectively on the natural rate of interest, and on consumption. The first one is based on an increase of labor taxes to sustain the natural rate of interest at an achievable level, and the second on reducing consumption taxes to sustain consumption at its full-employment level. Both approaches are equivalent given the budget constraint

of the Government, given by:

$$B_t^g = G_t - T_t + \frac{1 + r_{t-1}}{1 + g_{t-1}} B_{t-1}^g \quad (1.27)$$

Where total taxes per middle age household can be expressed as a function of output:

$$T_t = \tau_t^c C_t + \tau_t^l w_t L_t = \tau_t^c (Y_t - G_t) + \alpha \tau_t^l Y_t = (\tau_t^c + \alpha \tau_t^l) Y_t - \tau_t^c G_t \quad (1.28)$$

By replacing the above expression (1.28) in (1.27) we obtain an alternative expression for the budget constraint connecting the fiscal instruments. We call this equation the *Government budget rule*:

$$\frac{G_t}{Y_t} = \frac{\alpha \tau_t^l + \tau_t^c}{1 + \tau_t^c} + \frac{1}{Y_t} \left[B_t^g - \frac{1 + r_{t-1}}{1 + g_{t-1}} B_{t-1}^g \right] \quad (1.29)$$

with a steady state expression given by:

$$\frac{G}{Y} = \frac{\alpha \tau^l + \tau^c}{1 + \tau^c} + \frac{B^g}{Y} \left[\frac{g - r}{1 + g} \right] \quad (1.30)$$

Imagine there is a worsening of the credit constraints in this economy. If the Government is prevented to increase spending as well as public debt, then, in order to sustain the real interest rate at the same level, savings would have to contract through an increase of labor taxation on the middle age. The *Government budget rule* would then imply a reduction of the consumption tax, which would be the same required to sustain consumption at the same level, if the real interest rate would remain unchanged.

1.3 Counteracting a stable recession with distortionary taxes

In this section we introduce the notion of persistent recession based on the *Secular Stagnation* model proposed by Eggertsson and Mehrotra [21], and inspect the role of fiscal policy based on distortionary taxation to keep (or to bring back) the economy to a full-employment equilibrium. In order to prevent the economy to reach its first best full-employment allocation we introduce nominal prices in the model, and the possibility of a binding zero lower bound for the nominal interest rate that can prevent a negative natural rate of interest to be achieved, thus triggering a recession. We derive a simple sufficient condition to maintain a first best equilibrium possible, by ensuring consistency between monetary policy targets and distortionary taxation based fiscal policy. Then we introduce nominal wage rigidities that allow the appearance of a second stable equilibrium in the model, in this case a persistent recession. We revisit the previous sufficient condition to sustain the economy at its first best allocation, and also inspect how an adequate use of distortionary taxes, in particular a labor tax on firms, can help the economy move from a slump to a full-employment equilibrium.

1.3.1 Monetary and fiscal policy consistency

The maximization problem with nominal prices is given by:

$$\max_{C_t^y, C_{t+1}^m, C_{t+2}^o} E_t \{ \log(C_t^y) + \beta \log(C_{t+1}^m) + \beta^2 \log(C_{t+2}^o) \} \quad (1.31)$$

$$\text{s.t. } P_t(1 + \tau_t^c)C_t^y = P_t B_t^y \quad (1.32)$$

$$P_{t+1}(1 + \tau_{t+1}^c)C_{t+1}^m = Z_{t+1} + W_{t+1}L_{t+1}(1 - \tau_{t+1}^l) - (1 + i_t)P_t B_t^y + P_{t+1} B_{t+1}^m \quad (1.33)$$

$$P_{t+2}(1 + \tau_{t+2}^c)C_{t+2}^o = -(1 + i_{t+1})P_{t+1} B_{t+1}^m \quad (1.34)$$

And an exogenous borrowing limit: $(1+i_t)P_t B_t^y \leq P_{t+1} D_t$. Using the Fisher equation $1+i_t = (1+r_t)\frac{P_{t+1}}{P_t}$, the Euler Equation has the same expression as before, as well as the expressions for the real interest rate, and for the *Government budget rule*. The Euler Equation, in particular, is given by:

$$C_t^m = \frac{1}{\beta} E_t \left[C_{t+1}^o \frac{(1+\tau_{t+1}^c)}{(1+\tau_t^c)} \frac{P_{t+1}}{P_t} \frac{1}{1+i_t} \right] = \frac{1}{\beta} E_t \left[\frac{(1+\tau_{t+1}^c)}{(1+\tau_t^c)} \frac{C_{t+1}^o}{1+r_t} \right] \quad (1.35)$$

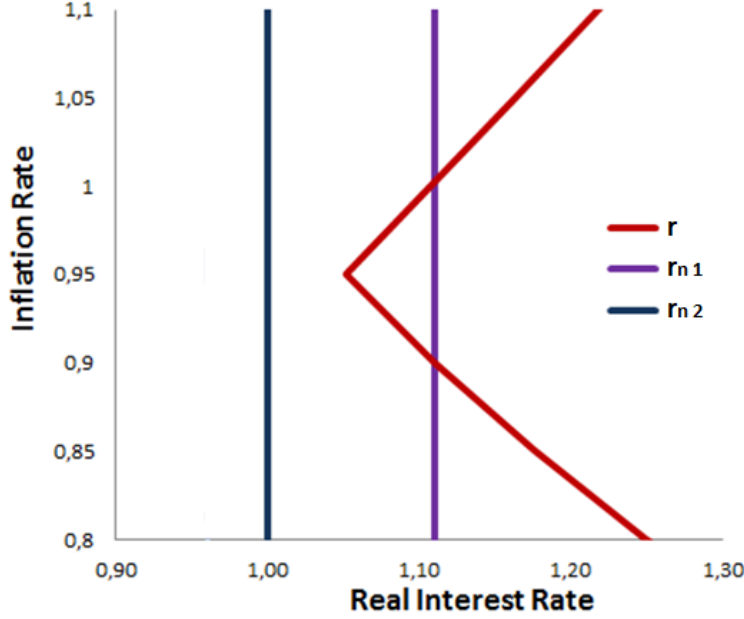
Assuming that the nominal interest rate follows a Taylor rule, $1+i_t = \max\{1, (1+i^*)\left(\frac{\Pi_t}{\Pi^*}\right)^{\phi_\pi-1}\}$, where $\phi_\pi > 1$, then by using the Fisher equation we get the following expression for the real interest rate:

$$1+r_t = \frac{1+i_t}{\Pi_t} = \max\left\{\frac{1}{\Pi_t}, \frac{1+i^*}{\Pi^*} \left(\frac{\Pi_t}{\Pi^*}\right)^{\phi_\pi-1}\right\} \quad (1.36)$$

Where Π^* and i^* are the monetary policy targets for gross inflation and the nominal interest rate. Note that when the zero lower bound is binding, the real interest rate is just a function of the current inflation level, and does not depend on any monetary policy instrument given by inflation and nominal interest rate targets. From the previous expression, and given the nominal interest rate zero lower bound, as well as this specific Taylor rule, a real interest rate lower bound is given by:

$$1+r_t \geq \frac{(1+i^*)^{\frac{1}{\phi_\pi}}}{\Pi^*} = 1+r^{kink} \quad (1.37)$$

If the natural rate of interest r_t^n is lower than the real interest rate lower bound then it would not be achievable, and a first best full-employment allocation would not be possible (see figure 1.1). r^{kink} then works like a lower bound for the natural rate of interest to allow a full-employment equilibrium in this economy. Another sufficient condition to ensure that the natural rate of interest is achievable is to use the implicit monetary policy target real interest rate level r^* as a lower bound for achievable levels of the natural rate of interest r_t^n . Let $1+r^* = \frac{1+i^*}{\Pi^*}$ be the implicit

Figure 1.1: *The Natural rate of Interest and the Zero Lower Bound*


monetary policy target real interest rate. Assuming that $\phi_\pi \geq 1$ then $r^* \geq r^{kink}$:

$$1 + r^{kink} = \frac{(1 + i^*)^{\frac{1}{\phi_\pi}}}{\Pi^*} = \frac{1 + i^*}{\Pi^*} (1 + i^*)^{\frac{1}{\phi_\pi} - 1} \leq 1 + r^*, \text{ for } i^* \geq 0, \phi_\pi \geq 1 \quad (1.38)$$

If the natural rate of interest r_t^n , determined by and adequate fiscal policy selection of distortionary tax levels, is greater than the monetary policy implicit real interest rate target $r^* = \frac{1+i^*}{\Pi^*}$, then a full-employment equilibrium is possible. In other words, fiscal and monetary policy are complements to fulfill full-employment conditions.

From an aggregate demand perspective we can reach the same conclusion about consistency between monetary and fiscal policy to allow first-best full-employment allocations. Note that aggregate demand $Y_t^d(\Pi_t)$ can be expressed by:

$$\Pi_t > \Pi^{kink}: Y_t^d = \frac{1}{1 + \tau_t^c} \left[\frac{(1 + \beta)(1 + g_t)D_t}{\beta} \frac{\Pi_t^{\phi_\pi}}{\Pi_t^{\phi_\pi - 1}} + D_{t-1} \right] + G_t \quad (1.39)$$

$$\Pi_t \leq \Pi^{kink}: Y_t^d = \frac{1}{1 + \tau_t^c} \left[\frac{(1 + \beta)(1 + g_t)D_t}{\beta} \Pi_t + D_{t-1} \right] + G_t \quad (1.40)$$

Where $\Pi_{kink} = \frac{\Pi^*}{(1+i^*)^{\frac{1}{\phi_\pi}}} \leq \Pi^*$. Also, $Y^{kink} = Y(\Pi^{kink})$ is the upper bound of the set

of admissible aggregate demand levels. Then, if the natural rate of interest has an achievable level, $r_t^n(\{\tau_t\}) \geq r^{kink}(\Pi^*, i^*)$, then $Y_t^d(\{\tau_t\}) = Y_t^f \leq Y^{kink}(\Pi^*, i^*)$, and a full employment is allowed in this economy. Note that aggregate demand expands with lower consumption taxes, an intuitive statement in this credit constrained environment. Note in addition that if inflation is above Π^{kink} , then aggregate demand Y_t^d is a negative function of inflation, and depends on monetary policy targets. Aggregate demand is a negative function of inflation because it is a negative function of the real interest rate ¹⁰, and the real interest rate in this case increases with inflation by means of the Taylor Rule as seen above. Otherwise, if inflation is below Π^{kink} then aggregate demand becomes a positive function of inflation, since in this case the gross real interest rate is the inverse of gross inflation. More deflation means higher real interest rates and lower demand. Furthermore, the lower segment of aggregate demand does not depend on any monetary policy instrument.

An example: sustaining the economy at full-employment

Imagine that for a given fiscal policy $\{\tau_h\}$ and a binding borrowing limit D_h the natural rate of interest $r_t^n(\{\tau_h\}, D_h)$ is achievable, such that $r_t^n(\{\tau_h\}, D_h) \equiv r_h^n \geq r^* = r(\Pi^*, i^*)$, where r^* is an implicit monetary policy real interest rate target. Let's further assume that $r_h^n = r^*$. If a deleveraging shock occurs and D falls from D_h to D_l , then to maintain the economy on a full-employment steady state it is sufficient to find a fiscal policy $\{\tau_l\}$ so that the resulting natural rate of interest is sustained at the same level equal to the monetary target real interest rate.

Then from expression (1.21) and (1.46) a full employment steady state equilibrium is sustained if:

$$r_l^n = r_h^n = r^* \iff \frac{1 + \tau_l^c}{1 + \tau_h^c} = \frac{D_l}{D_h} = \frac{1 - \alpha\tau_l^l}{1 - \alpha\tau_h^l} \quad (1.41)$$

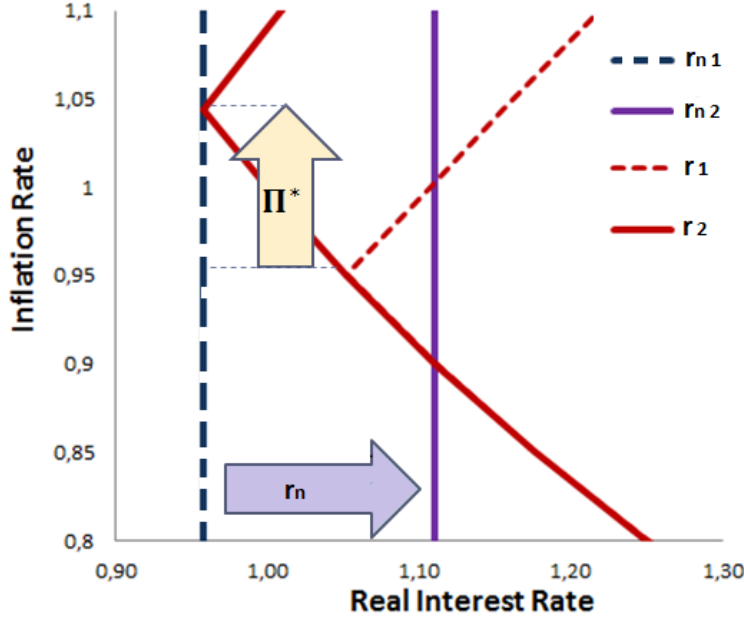
¹⁰Given the same assumptions for the elasticity of intertemporal substitution, excess savings is positively sloped with respect to the real interest rate, and consequently current demand is negatively sloped with respect to the same variable.

Which is equivalent to:

$$\partial r^n = 0 \iff \partial \tau^l = -\frac{\partial D}{D_h} \left(\frac{1}{\alpha} - \tau_h^l \right) \text{ and } \partial \tau^c = \frac{\partial D}{D_h} (1 + \tau_h^c) \quad (1.42)$$

If $\tau_h^c = \tau_h^l = 20\%$ and labor share $\alpha = 0.7$, then a 5% reduction of D would have to be offset by an increase of the labor tax from 20% to 26%, together with a reduction of consumption tax from 20% to 14%. A combination of fiscal and monetary policy may attenuate the fiscal effort needed to offset a deleveraging shock in order to sustain employment (see figure 1.2). But only fiscal policy may prevent the appearance of

Figure 1.2: Allowing a First Best Solution with Fiscal and Monetary Policy



a secular stagnation steady state to where the economy can be dragged if fiscal and monetary policy agents are not assertive enough. We will see how next, when wages are sticky.

1.3.2 Sticky wages, stable recessions, and labor tax on firms

We now introduce in the model sticky wages and a wage tax on firms τ^w . the tax works similarly to the labor tax on households in the way it reduces their net income

and savings, having the same kind of impact on the natural rate of interest. But τ^w may also play a relevant role to avoid or to leave a *secular stagnation* in the presence of wage rigidities. The firm problem is now given by:

$$Z_t = \max_{L_t} P_t Y_t - W_t L_t^d (1 + \tau_t^w) \text{ s.t. } Y_t = A_t (L_t^d)^\alpha \quad (1.43)$$

We continue to assume for simplicity that $A_t = A = 1$. The solutions for $w_t = \frac{W_t}{P_t}$ and $z_t = \frac{Z_t}{P_t}$ are given by:

$$w_t = \frac{\alpha}{1 + \tau_t^w} \frac{Y_t}{L_t} \text{ and } z_t = (1 - \alpha) Y_t \quad (1.44)$$

While the aggregate demand expression given by (1.39) and (1.40) remains unchanged with the introduction of this tax, close-form expressions for the real interest rate and the *Government budget rule* are given by:

$$1 + r_t = \left(\frac{1 + \beta}{\beta} \right) \frac{(1 + g_t) D_t}{(1 - \alpha \tau_t^{lw}) Y_t - D_{t-1}} \quad (1.45)$$

$$\frac{G_t}{Y_t} = \frac{\alpha \tau_t^{lw} + \tau_t^c}{1 + \tau_t^c} \quad (1.46)$$

where $\tau_t^{lw} = \frac{\tau_t^l + \tau_t^w}{1 + \tau_t^w}$ is a labor tax index combining τ^l and τ^w in a single expression, which besides acting similarly to τ_t^l in sustaining the natural rate of interest, can play a relevant role in leaving a *Secular Stagnation* by interfering directly on the aggregate supply side of this economy, that we describe next.

A wage rigidity is introduced in the model such that households will not accept working for a wage lower than a nominal wage norm \tilde{W} given by:

$$\tilde{W}_t = W_{t-1}^\gamma \left(P_t w_t^{flex} \right)^{1-\gamma} \quad (1.47)$$

whereas the nominal wage will always be greater or equal than the flexible labor

full-employment nominal wage:

$$W_t = \max\{\tilde{W}_t, P_t w_t^{flex}\} \quad (1.48)$$

With an equivalent expression for aggregate supply given by:

$$Y_t = \min \left\{ Y^f, Y^f \left(\Pi_t \frac{1 + \tau_{t-1}^w}{1 + \tau_t^w} \right)^{\gamma \frac{\alpha}{1-\alpha}} \left(\frac{Y_{t-1}}{Y^f} \right)^\gamma \right\} \quad (1.49)$$

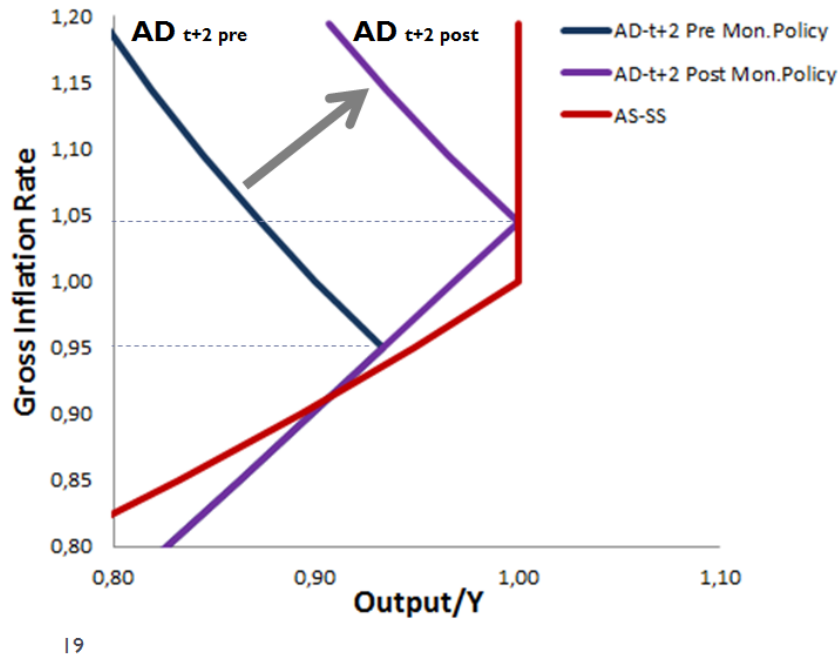
Note that, from the wage equilibrium equation (1.44), an increase of the labor tax on firms triggers a downward pressure on the nominal wage to sustain firms profits. But if the full-employment nominal wage cannot be fully achieved because of the nominal rigidity, then supply will adjust to a sub-employment level. The corresponding steady state expression is given by:

$$Y = Y^f \min \left\{ 1, \Pi^{\frac{\gamma}{1-\gamma} \frac{\alpha}{1-\alpha}} \right\} \quad (1.50)$$

which is an increasing function of inflation for negative inflation levels, and does not depend on any fiscal instrument, assuming that a perpetual constant change of the labor tax on firms, although theoretically viable, is unrealistic. Note that the steady state expression for aggregate supply does not depend on any monetary policy target either. And neither the lower segment of aggregate demand given by expression (1.40).

Figure 1.3 shows a graphical representation of a *Secular Stagnation* equilibrium, determined by the intersection of the lower segments of aggregate demand and supply. This equilibrium is stable, as described in detail by Eggertsson and Mehrotra [21], by assuming that the wage rigidity is sufficiently high in order to ensure that the slope of the lower segment of aggregate supply is smaller than the slope of lower aggregate demand. Figure 1.3 also shows that monetary policy cannot neutralize a stable recession, although it can provide the economy with a full-employment equi-

Figure 1.3: *Secular Stagnation* Equilibrium



librium by allowing an achievable negative natural rate of interest with an adequate positive inflation target.

In fact, only fiscal policy is able to neutralize a stable recession in this economy. The fiscal mechanism to counteract a *Secular Stagnation* equilibrium needs to boost inflation, since any stable recession is characterized by an equilibrium in deflation. This inflation boost can be achieved by fiscally expanding demand, contracting supply, or a combination of both.

i) Aggregate Demand expansion to counteract a stable recession

Aggregate supply in steady state does not depend on any fiscal instrument, then only a sufficiently assertive permanent demand expansion can clear a recession from the set of available equilibria, thus forcing a permanent transition to a stable full-employment state.

Graphically it is easy to see that a demand expansion must be big enough so that the lower segment of aggregate demand does not intersect aggregate supply, which

is the same to say that the lower segment of aggregate demand must achieve a full-employment level for an inflation level smaller than zero. A sufficient condition is to ensure that the natural rate of interest becomes positive and achievable, corresponding to a negative inflation intersection of the lower aggregate demand segment with full-employment output. In addition, a negative natural rate of interest in this economy implies the existence of a *Secular Stagnation* equilibrium¹¹. The sufficient aggregate demand expansion is achieved by a reduction of the consumption tax given by¹²:

$$\partial\tau^c = (1 + \tau_{ss}^c) \left[\left(\frac{Y_{ss} - G}{Y^f - G} \right) \left(\frac{\Pi_{ss} + \frac{\beta}{(1+\beta)(1+g)}}{1 + \frac{\beta}{(1+\beta)(1+g)}} \right) - 1 \right] < 0 \quad (1.51)$$

The sufficiency of this result presumes that the resulting natural rate of interest level r^n is achievable from a monetary policy standpoint, or $r^* \leq r^n$.

ii) Aggregate Supply contraction to counteract a stable recession: Paradox of Toil

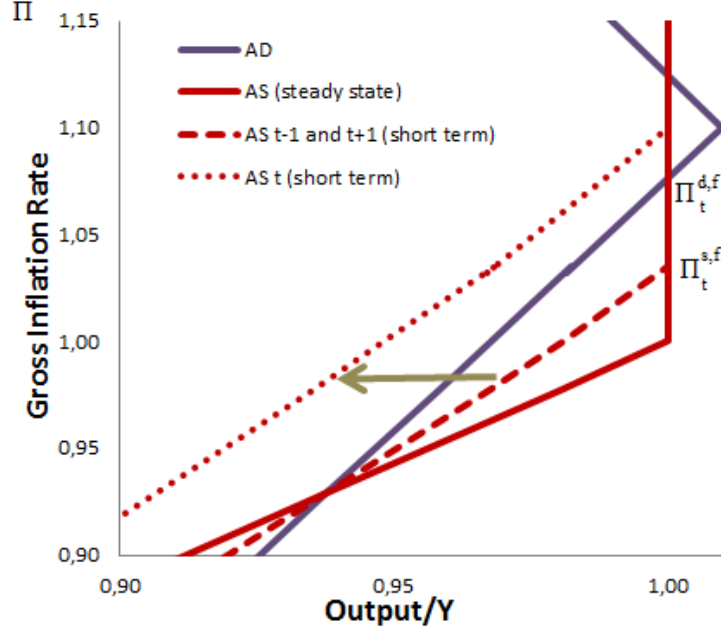
A negative natural rate of interest implies the existence of a *Secular Stagnation* equilibrium in the model. But a full-employment equilibrium may coexist if the natural rate of interest is achievable by an adequate monetary policy such that $1 + r^* < 1 + r^n$. From Figure(1.4) we can observe that, in this model a necessary and sufficient condition for the existence of a secular stagnation equilibrium is that (i) the lower segment of aggregate demand is steeper than the lower segment of aggregate supply[21], and that (ii) the inflation level corresponding to short term aggregate supply determined at full employment, $\Pi_t^{s,f}$, is lower than the inflation level corresponding to aggregate demand determined at full employment¹³, $\Pi_t^{d,f} = \frac{1}{1+r_t^n} \cdot \Pi_t^{s,f}$ is determined by equation (1.49), and the existing condition for a secular

¹¹as the lower segment of aggregate demand intersects full employment output at a positive inflation level, which by construction of the model ensures an intersection with the lower segment of aggregate supply.

¹²where the subscript *ss* means *Secular Stagnation steady state equilibrium*, and *f* means *full-employment steady state*.

¹³ $\Pi_t^{d,f}$ is by construction equal to the inverse of the gross natural rate of interest.

Figure 1.4: One period aggregate supply contraction shock



stagnation is given by¹⁴:

$$\Pi_t^{s,f} = \left(\frac{1 + \tau_t^w}{1 + \tau_{t-1}^w} \right) \left(\frac{Y^f}{Y_{t-1}} \right)^{\frac{1-\alpha}{\alpha}} < \frac{1}{1 + r_t^n} = \Pi_t^{d,f} \quad (1.52)$$

It is possible to force a transition from a stable recession to a stable full-employment equilibrium, with a transitory contraction of aggregate supply, enough to clear at least for one period the possibility of a *Secular Stagnation* equilibrium (Figure 1.4). Using a similar mechanism as the previous one for aggregate demand, it is sufficient to contract aggregate supply by increasing the labor tax on firms, such that the lower segment of aggregate supply, given by expression (1.49), intersects full-employment output at a gross inflation level greater than the intersection of the lower aggregate demand segment with full-employment output, given by a gross inflation level equal to the inverse of the gross natural rate of interest $\frac{1}{1+r_t^n}$. This condition is directly

¹⁴Note that if the labor tax on firms is unchanged from previous period, by construction in a secular stagnation equilibrium, $\left(\frac{Y^f}{Y_{t-1}} \right)^{\frac{1-\alpha}{\alpha}} < \frac{1}{1+r_t^n} \Leftrightarrow \frac{1}{1+r_t^n} \left(\frac{Y_{t-1}}{Y^f} \right)^{\frac{1-\alpha}{\alpha}} > 1$.

derived from expression(1.52):

$$\frac{1 + \tau_t^w}{1 + \tau_{t-1}^w} \geq \frac{1}{1 + r_t^n} \left(\frac{Y_{t-1}}{Y^f} \right)^{\frac{1-\alpha}{\alpha}} > 1 \quad (1.53)$$

Note also that the natural rate of interest may remain constant if the increase of labor tax on firms¹⁵ is compensated from a budget perspective by a reduction of the labor tax on households so that the consumption tax remains unchanged.

Similarly to Correia et al. [16] a liquidity trap is resolved with an inflation boost, bringing back the economy to a full-employment first best equilibrium from a stable recession, but with a reverse usage of distortionary taxes in this credit constrained closed economy environment.

iii) A fiscal rule to prevent a secular stagnation

Note that the previous condition (1.53) can be used as a *fiscal rule* to prevent an economy to fall into a persistent recession, when a full-employment equilibrium and a *secular stagnation* recessions are, both, achievable steady state equilibria. When monetary policy is ineffective this *fiscal rule* would work like a *Taylor rule*, but using fiscal, instead of monetary instruments. In our economy, a *secular stagnation* equilibrium given by $Y_t < Y^f$, is driven by a stable deflation steady state where, from expression (1.49), output at time t is expressed by:

$$Y_t = Y^f \Pi_t^{\gamma \frac{\alpha}{1-\alpha}} < Y^f \text{ if } \Pi_t < 1 \quad (1.54)$$

Combining the previous expression with the condition given by equation (1.53), our

¹⁵Note also that the existence of a secular stagnation equilibrium implies that the left-hand side of the equation is greater than one.

fiscal rule can then be expressed by:

$$\Pi_t \geq 1 \Rightarrow \tau_t^w \leq \tau_{t-1}^w \quad (1.55)$$

$$\Pi_t < 1 \Rightarrow \tau_t^w = (1 + \tau_{t-1}^w) \frac{\Pi_t^\gamma}{1 + r_t^n} - 1 > \tau_{t-1}^w \quad (1.56)$$

This means that the government commits to increase the labor tax on firms if inflation becomes negative, in order to ensure that real wages do not increase above the flexible wage level, thus sustaining full employment, all in all assuming that the *natural rate of interest* is known and achievable.

iv) Welfare implications

The two main mechanisms to move our economy out of a slump may be welfare improving for all agents. An aggregate demand expansion via a consumption tax reduction may clear the secular stagnation equilibrium, altogether not requiring an increase of labor taxes from a Government budget equilibrium standpoint. From the Government budget constraint given by equation(1.27), a sufficient decrease in consumption taxes can increase aggregate demand to a full-employment level without the need to reduce labor taxes. If the natural rate of interest in this economy is achievable, then the three types of agents would be better off.

The same result may be obtained using the fiscal rule, if the increase of the labor tax on firms is combined with a reduction of the labor tax on households, in order to sustain the labor tax index τ_t^{lw} , as well as the natural rate of interest which we assume achievable, at the same level¹⁶.

¹⁶Nevertheless, if the natural rate of interest must be increased to be achievable (or sustained at an achievable level during a credit shock), then the combination of increasing labor taxes with decreasing consumption taxes is welfare improving for the old that benefit from the consumption tax reduction. The welfare implications for the young and middle age should be further inspected.

1.3.3 Return on Capital Income tax

Introducing capital and tax on capital in the model does not qualitatively change the fiscal policy options to counteract a persistent recession we have derived until that point. But by having an impact similar to the consumption tax in expanding aggregate demand, reducing the tax on capital can be an effective alternative to a reduction of consumption taxes when further promoting a consumption expansion is not a desirable outcome. Although reducing the tax on capital has also an expanding impact on aggregate supply, the impact on demand prevails, so that the net effect is an increase of the natural rate of interest, and of equilibrium inflation.

We use a version with capital of the previous model[21] with a tax on capital, together with all the other taxes already presented in this chapter. We introduce distortionary taxes including the tax on capital income in the budget constraints of the middle age and old, which are now given by:

$$(1 + \tau_{t+1}^c)C_{t+1}^m = z_{t+1} + w_{t+1}L_{t+1}(1 - \tau_{t+1}^l) + K_{t+1}[r_{t+1}^k(1 - \tau_{t+1}^k) - 1] - (1 + r_t)B_t^y + B_{t+1}^m \quad (1.57)$$

$$(1 + \tau_{t+2}^c)C_{t+2}^o = -(1 + r_{t+1})B_{t+1}^m + K_{t+1}(1 - \delta) \quad (1.58)$$

and the labor tax on firms is still considered in the firm problem, now given by:

$$Z_t = \max_{L_t, K_t} \{P_t Y_t - W_t L_t(1 + \tau_t^w) - P_t r_t^k K_t\} \text{ s.t. } Y_t = A_t L_t^\alpha K_t^{1-\alpha} \quad (1.59)$$

where, $w_t = \frac{W_t}{P_t} = \frac{\alpha L_t^{\alpha-1}}{1 + \tau_t^w} = \frac{\alpha}{1 + \tau_t^w} \frac{Y_t}{L_t}$ and $r_t^k = (1 - \alpha)A_t L_t^\alpha K_t^{-\alpha} = (1 - \alpha)\frac{Y_t}{K_t}$. From the return on capital expression we can directly observe that a reduction of the tax on capital income reduces the cost of capital:

$$r_t^k = \frac{1}{1 - \tau_t^k} \left(1 - \frac{1 - \delta}{1 + r_t} \right) \quad (1.60)$$

Aggregate demand expands when if the tax on capital income decreases. This can be

directly observed from the following expression of aggregate demand in real terms:

$$Y_t^d = \frac{1}{1 - \alpha\tau_t^{lw} - (1 - \alpha)B_t^l} \left[\frac{1 + \beta}{\beta} \frac{(1 + g_t)D_t}{1 + r_t} - D_{t-1} \right] \quad (1.61)$$

Where $B_t^l = \frac{1}{r_t^k} \frac{1+\beta}{\beta} \frac{1-\delta}{r_t+\delta} = (1 - \tau_t^k) \frac{1+\beta}{\beta} \frac{1-\delta}{r_t+\delta}$.

Regarding aggregate supply, its expressions is equal to the one derived previously and given by expression (1.49). But now full employment output is given by:

$$Y_t^f = A_t \bar{L}^\alpha K_t^{1-\alpha} = \bar{L} A_t^{\frac{1}{\alpha}} \left(\frac{1 - \alpha}{r_t^k} \right)^{\frac{1-\alpha}{\alpha}} = \bar{L} A_t^{\frac{1-\alpha}{\alpha}} \left[\frac{(1 - \alpha)(1 - \tau_t^k)}{1 - \frac{1-\delta}{1+r_t}} \right]^{\frac{1-\alpha}{\alpha}} \quad (1.62)$$

The difference lies on the aggregate supply expression for positive inflation levels which is not constant, and expands when τ^k decreases, leading also to an aggregate supply expansion when inflation is negative. Although a reduction of the tax on capital income has an expanding impact on both aggregate supply and demand, the impact on demand prevails. The resulting impact on inflation, employment and the natural rate of interest are qualitatively similar to the ones derived for the consumption tax, being an available adequate alternative to this instrument in counteracting a persistent recession.

1.4 Final remarks

In this Chapter we formalized the role of distortionary taxation in avoiding a stable recession characterized by the *Secular Stagnation* framework proposed by Eggertsson and Mehrotra [21], based on a three generations OLG model with borrowing constraints. We compare our results with the ones obtained by Correia et al. [16] that propose a solution based on the same set of distortionary taxes in a standard single agent New Keynesian model without borrowing constraints, to counteract a liquidity trap that is by construction temporary. We find reversed results. Our

mechanism is based on a wealth re-distributive policy using distortionary taxes to increase the natural rate of interest so that it becomes achievable given the monetary policy targets, by increasing labor taxes on the middle age employed and redistributing the tax proceeds to the population in general by reducing consumption tax. Instead, Correia et al. [16] emulate inflation in consumer prices using an increasing path of consumption taxes, so that the intertemporal condition allows an achievable negative natural rate of interest, and the liquidity trap is neutralized. We use the same fiscal toolbox with different approaches. We increase the natural rate of interest to a level consistent with monetary policy effectiveness, instead of temporarily allowing a first best solution compatible with a negative natural rate of interest.

As an alternative to the standard fiscal policy prescriptions to counteract an economic downturn, based on Keynesian increases of public expenditures, more public debt and tax cuts to stimulate demand¹⁷, and complementing the papers referenced above, the main purpose of our analysis is to show how fiscal policy based on distortionary taxation can be effective in avoiding persistent recessions, in particular when increasing public expenditures and debt are not policy options available.

¹⁷Besides other non-fiscal approaches, namely the one proposed by Eggertsson and Woodford [23] where the central bank commits to keep interest rates at a lower level even after a recession resulting from a liquidity trap is over.

Chapter 2

Age Milestones and Low Interest Rates, an Analytic approach

ABSTRACT

Major age milestones like the age of first job, retirement age, or life expectancy, bounding relevant economic periods in a persons' life, have been changing substantially during the last decades. In parallel real interest rates have been significantly declining in relevant world economies, reaching stable negative levels in some cases. We propose an analytic approach to relate those two phenomena by using an overlapping multi-generations model to find expressions for real interest rate elasticities to age parameters. The model formalizes the mechanisms supporting the relation between interest rates and age, sheds light on the relative importance of each age milestone in explaining changes of real interest rates, and how other factors like elasticity of inter-temporal substitution, population and productivity growth, inter-generational altruism, as well as a social security system, may mitigate or amplify those changes.

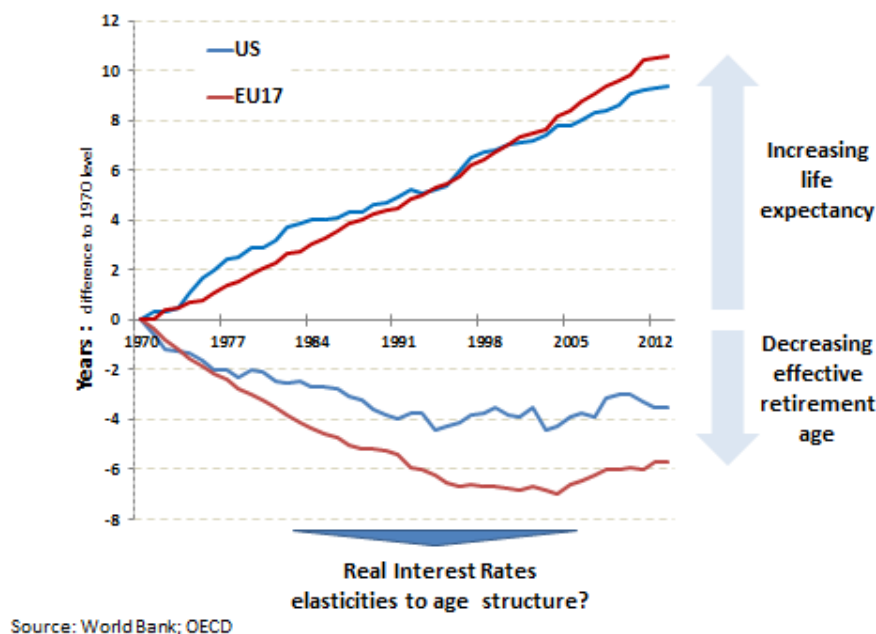
2.1 Introduction

During the last decades, the age structure of the population in some of World's most relevant economies has changed significantly. For example, although Life expectancy at birth increased by approximately ten years since the 70's both in US and EU, retirement age has declined four and six years respectively, contributing to raise the need to save in those economies (Figure 3.1). Furthermore, the recent economic crisis tended to affect the average age of first job as firms tend to postpone hiring as a way to adjust down employment level, which could lead to an increase of the borrowing needs of this population segment.

Changes in age milestones determine many aspects of relevant economic periods of a persons' life, which themselves may directly impact real interest rates through changes of borrowing and savings paths. For example, for a higher effective retirement age, people need to save less for their expected retirement period, leading to a contraction of savings and a consequent increase in equilibrium real interest rates. In addition, postponing the age of first job increases the duration of borrowing after adulthood, pushing interest rates upwards too. Increasing both parameters, age of retirement and first job, at the same time and by the same amount, although not changing the duration of the working period, may impact the real interest rate by affecting borrowing and saving paths, and consequently loan market equilibrium and real interest rates.

Although the impact of age structure in relevant World economies has been a recurrent topic covered in recent literature, in particular to explain the persistent decline of interest rates, economic stagnation and liquidity traps, there has not yet been an attempt, to the best of our knowledge, to formally derive the analytic relations of real interest rates with respect age milestones. The general omission of changing demographic parameters in most current formal economic models ignores a potentially relevant factor influencing equilibrium conditions, and consequently the type

Figure 2.1: Life Expectancy and effective Retirement Age: EU and US 1970-2014



and even sign of solutions. For example, an increase of life expectancy can drag the full-employment equilibrium real interest rate from positive to negative. Since a negative level may not be achievable when the nominal interest rate zero lower bound is binding, a first best solution may no more be available in such a model. The same can happen with operative bequest motives, which may become inoperative, for example if the retirement age decreases, or life expectancy increases.

The purpose of this paper is to fill out this gap. By merging an age structure framework with an OLG model, we derive tractable algebraic real interest rate elasticity expressions with respect to each age parameter, to shed light on the demographic formal mechanisms that influence real interest rates, and inspect in particular the examples mentioned above. Moreover we provide a straightforward alternative to heavy computational quantitative models, in order to illustrate the impact of demographic factors on general economic phenomena.

Ikeda and Saito [29] study the effects of demographic changes on the real interest rate in Japan by capturing demographic dynamics by exogenous changes of the ratio

of workers to total population. But most of the literature covering the present topic use *perpetual youth* type models inspired by Blanchard and Fischer [7], using transition probabilities between age groups. This approach, that facilitates aggregation of individual agents, thus ensuring analytically more tractable life-cycle models, was adopted, for example, by Carvalho and Ferrero [11] to explain Japan's persistent deflation, using transition probabilities from worker to retired, and from retired to death, and by Carvalho et al. [12] to inspect the mechanisms of how demographics affect real interest rates. Similarly, Aksoy et al. [2] relate macroeconomic trends to demographic structure with a model to which they add an additional transition probability from young to worker, after conducting an empirical study where they found evidence that differences in generation weight across countries explain differences among main macro-economic variables.

Nevertheless, transition probabilities in those models tend to be independent of age, and of time since transition from previous age groups, which makes them less appropriate to derive analytic relations between interest rate and explicit age milestones. This circumstance was recently overcome by Eggertsson and Robbins [22] who used a quantitative overlapping multi-generations model inspired by the work of Auerbach and Kotlikoff [3] to investigate the decline of real interest rates in US.

Similarly, we use an overlapping multi-generations model, where most relevant age milestones are exogenous parameters, allowing to analytically express the real interest rate in terms of age structure changes, surprisingly not affecting algebraic tractability, in order to shed light on relevant demographic mechanisms that are dragging down real interest rates.

In what follows, we begin by outlining an overlapping generations deterministic model in the context of an endowment economy, where agents are economically active after childhood until their age of life expectancy. The number of generations of the model depends already on those two age milestones. At the age of adulthood

agents start borrowing to consume. From the age of first job until retirement they receive an income in the form of an endowment, with which they pay back their debt, consume, and save for retirement¹. During that working period, at a certain moment in time agents have payed back their debts and start saving for retirement. Until that moment agents are borrowers, and after they become savers. The initial savings age is an endogenous variable of the model. During retirement they use their accumulated savings to consume. We derive the equilibrium conditions and aggregate expressions for the main variables of the model, in particular of excess borrowing, in terms of the real interest rate and age milestones, which becomes zero for loan market equilibrium.

In the third section we use the excess borrowing expression at loan market equilibrium in steady state to formalize the analytic relation between the *natural rate of interest*² and age structure. We formalize the derivatives of real interest rate with respect to each age parameter, using the partial derivatives of excess borrowing with respect to age milestones, and to the real interest rate. We find that excess borrowing decreases with increasing interest rates if the elasticity of inter-temporal substitution is above a certain acceptable threshold level that depends on the relative duration of retirement. We use this assumption throughout the paper, so that the consistent negative slope of excess borrowing with respect to the real interest rate allows the sign of age milestones elasticities to be determined by the signs of the partial derivative of excess borrowing with respect to each age parameter.

In the fourth section we introduce intergeneration transfers in the form of bequests to children, of gifts to parents, and of a *pay-as-you-go* social security system, as those concepts are closely related to agents age structure, in particular to the age when

¹In our model the age of *adulthood* and age of *first job* may be different. After adulthood and before the age of first job agents have to borrow in order to consume. In the special case where those two age milestones are set the same, the algebraic expressions are simplified, and the calibrated outputs are not materially different. An alternative not used in our model to date, would be to consider endogenous transfers from parents to children during that specific phase of their lives.

²The *natural rate of interest* is defined as the full-employment equilibrium real interest rate.

their children are born. We also analyze how intergeneration transfers parameters affect the elasticities of real interest rates with respect to age milestones. Finally, in section five we calibrate a model with endogenous output and capital to quantify the analytic results of previous sections. We also test the impact of changing capital depreciation on real interest rate elasticities with respect to age milestones, as well as the impact of changes in age structure on the capital-output ratio.

As we have already noted, this chapter focuses on the presentation of a framework that allows to derive formal algebraic relations between real interest rates and age milestones, in order to inspect the influence of demographic factors in specific economic mechanisms. In particular, we use our framework to explore how changing age structure can switch an altruistic motive from helping children to supporting parents, or how an increase of life-expectancy, a reduction of retirement age, and postponing of the age of first job explain the decline of real interest rates, further quantifying those phenomena. Although we keep our results focus on the demand side of OLG models, our framework can also be used, for example, to explore secular stagnation mechanisms driven by demographic factors. In particular, in current work in progress, we extend our framework with nominal prices, endogenous output, and nominal wage rigidities, where when the natural rate of interest becomes negative, a second best solution with a sub-optimal stable equilibrium output level is characterized by an endogenous persistent increase of the age of first job.

2.2 An Endowment Economy with age milestones

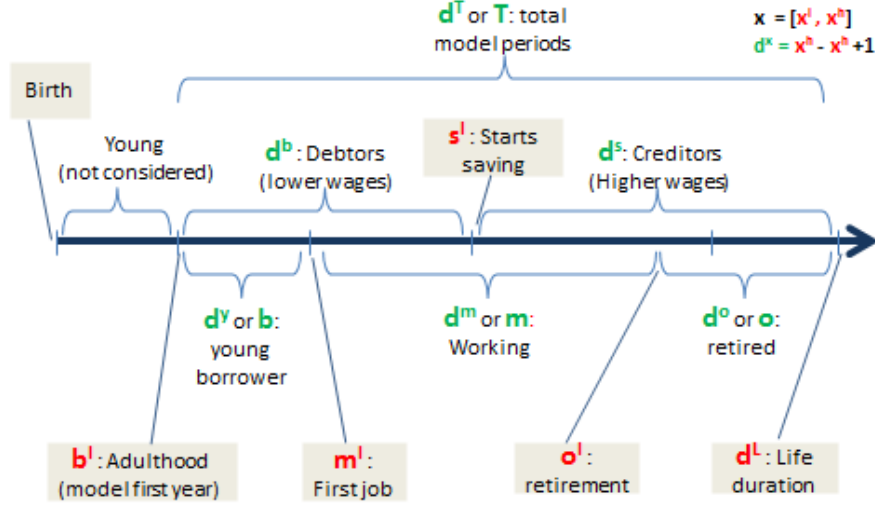
In this section we describe and solve a multi-generations OLG model where age milestones binding relevant economic periods of households, can exogenously change. We also derive some algebraic tools that simplify the model solution in closed-form expressions, and with which the derivatives of the steady state equilibrium real interest rate with respect to age milestones can be algebraically explicitly derived.

CHAPTER 2. AGE MILESTONES AND LOW INTEREST RATES

Consider an overlapping generations model in the spirit of Eggertsson and Mehrotra [21] where new generations start every year. Imagine that households live $L \equiv d^L$ years (where d^L stands for *duration of life*), but are considered economically active in the model only after childhood, from the age of adulthood b^l (b^l standing for *lower borrowing age*) until the last year of their lives at age d^L . The number of overlapping generations of the model $T = d^L - b^l + 1$ is then determined by two age milestones, bounding the period that starts at the age of adulthood, and ending at the last year of their lives. We start by considering an endowment economy where agents have no capital to invest in, but where households can lend to one another. After childhood, at age b^l households borrow from other households to consume. During the middle-age period m^l (m^l standing for *lower middle age*) they receive an income in the form of endowment $y_t^{i=age}$ which they use to consume, to pay-back their debts, and to save for retirement by lending to other households. In order to smooth their life-time consumption path, during the first part of their middle age period households are borrowers, becoming savers thereafter until the end of their lives. The *initial saving age* $s^l \in [m^l, o^l[$ is an endogenous parameter of the model. Households are retired from age o^l to d^L , having no endowment and consuming with the proceeds from their savings during that period.

The model age structure is illustrated in Figure 2.2. Age milestones in red are the boundaries of life economic periods with durations in green. We can look to an household from an income perspective, starting his journey as a *young borrower* without income who needs to borrow from other agents to be able to consume. The *young borrower's* period has a duration in years of $d^b = m^l - b^l$. He then enters into *middle age*, with a duration in years of $d^m = o^l - m^l$, after finding his first job at age m^l , and gets an income in the form of endowment until retirement at age o^l . Thereafter he will be retired for $d^o = T - d^m - d^b = L - m^h$ years. Alternatively we can look to an household from a borrowing/saving perspective, which may facilitate the economic intuition: in the beginning of their journey they are net *borrowers*

Figure 2.2: Relevant Life-cycle Periods and Age Milestones



during d^b years until they pay back their loans, and become savers at the age s^l for $d^s = d^L - s^l + 1$ years. It is the relative weight of borrowers and savers, or the interaction between loan demand and supply, that will determine loan market equilibrium interest rate level. Note that the initial saving age s^l , which determines the relative weight of borrowers and savers, is an endogenous parameter of the model. s^l itself depends on the relative duration of *young borrowers*, middle age and *retirement* periods, with durations respectively given by b , m , and o respectively. In what follows we will use durations notation b , m and model life span $T = b + m + o$ to express most of our findings, where $b = m^l - b^l = b^h - b^l + 1$, $m = m^h - m^l + 1$, and o is retirement duration, here a dependent variable.

Consider then a representative household reaching adulthood at time t , with the following utility function:

$$\max_{c_{t+i}^{b^l+i}} E_t \sum_{i=0}^{T-1} \beta^i U(c_{t+i}^{b^l+i}) \quad (2.1)$$

Where the $U(c)$ is assumed to be a constant elasticity of inter-temporal substitution

CHAPTER 2. AGE MILESTONES AND LOW INTEREST RATES

utility function expressed by $U(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$. $c_{t+i}^{b^l+i}$ is the consumption of households with age $b^l + i$ at time $t + i$. Furthermore agents borrow and lend to one-another using one year risk-free bonds at an interest rate r_t . We can then write the annual budget constraints faced by an agent reaching adulthood at time t , for the rest of his life.

$$c_t^{b^l} = y_t^{b^l} + b_t^{b^l} \quad \text{for age} = b^l \quad (2.2)$$

$$c_{t+i}^{b^l+i} = y_{t+i}^{b^l+i} + b_{t+i}^{b^l+i} - (1 + r_{t+i-1})b_{t+i-1}^{b^l+i-1} \quad \text{for age } b^l + i \in]b^l, d^L[\quad (2.3)$$

$$c_{t+T-1}^{d^L} = y_{t+T-1}^{d^L} - (1 + r_{t+T-2})b_{t+T-2}^{d^L-1} \quad \text{for age} = d^L \quad (2.4)$$

The three budget constraints are similar and can be analyzed from two perspectives. Considering equation (2.3), an agent with age $b^l + i$ at time $t + i$ takes his endowment $y_{t+i}^{b^l+i}$ together with a new loan $b_{t+i}^{b^l+i} > 0$ to consume $c_{t+i}^{b^l+i}$ and pay his loan and interest corresponding to the previous period $(1 + r_{t+i-1})b_{t+i-1}^{b^l+i-1}$. Alternatively an agent takes his endowment $y_{t+i}^{b^l+i}$ together with his savings $-(1 + r_{t+i-1})b_{t+i-1}^{b^l+i-1}$ from previous year to consume $c_{t+i}^{b^l+i}$ and to save $b_{t+i}^{b^l+i}$ for the next year. In this case $b_{t+i}^{b^l+i}$ is negative. At the age of adulthood (2.2) there are no loans to pay back from previous year. During the last year of their lives, (2.4), households do not need to save for the future any longer. Let:

$$y_t^i > 0 \text{ for } i \in [m^l, m^h] \forall t \quad (2.5)$$

$$y_t^i = 0 \text{ for } i \in [b^l, m^l[\cup]m^h, d^L] \forall t \quad (2.6)$$

Although endowments are assumed to be strictly positive for the middle age and zero otherwise - equations(2.5) and (2.6) - we do not need to impose any special restriction on the duration of the middle age period in order to have distinct borrowing and savings periods. In particular the duration of middle age period $m = d^m$ could coincide with model's time span T . It is the endowment and real interest rate paths

that will determine the annual path of agent loans $b_{t+i}^{b^l+i}$ that can be positive or negative in order to smooth households' consumption path, and consequently of aggregate loans throughout all living households, that from now on we define as *excess borrowing*:

$$B_t^{b^l} = \frac{1}{N_t^{b^l}} \sum_{i=0}^{T-1} N_t^{b^l+i} b_t^{b^l+i} = \sum_{i=0}^{T-1} \frac{N_t^{b^l+i}}{N_t^{b^l}} b_t^{b^l+i} = \sum_{i=0}^{T-1} \frac{b_t^{b^l+i}}{\prod_{k=0}^{i-1} (1 + g_{t-k})} \quad (2.7)$$

where $B_t^{b^l}$ is the value of excess borrowing at time t normalized to the size of the younger generation with age b^l at time t , and $1 + g_t = \frac{N_t^{b^l}}{N_{t-1}^{b^l}} = \frac{N_t^{b^l}}{N_t^{b^l+1}}$ is population growth at time t . When excess borrowing is equal to zero then the loan market is in equilibrium. An equilibrium real interest rate solution path ensures that loan market is in equilibrium at any time t . If we can find a closed form expression (at least for steady state equilibria) for excess borrowing as a function of the real interest rate r and other exogenous parameters of the model x , then we could use loan market equilibrium equation $B_t^{b^l}(r, x) = 0$ and the implicit function theorem to find closed form expressions for changes of the equilibrium real interest rate with respect to changes of any parameter of the model x , in particular age milestones:

$$B_t^{b^l}(r, x) = 0 \Rightarrow r_x \equiv \frac{\partial r}{\partial x}(r, x) = -\frac{\frac{\partial B_t^{b^l}}{\partial x}(r, x)}{\frac{\partial B_t^{b^l}}{\partial r}(r, x)} \Leftrightarrow r_x = -\frac{B_x}{B_r}(r, x) \quad (2.8)$$

Excess borrowing at time t can be expressed³ as a function of aggregate endowment and consumption at time t , and excess borrowing at time $t - 1$:

$$B_t^{b^l} = C_t^{b^l} - Y_t^{b^l} + \left(\frac{1 + r_{t-1}}{1 + g_t} \right) B_{t-1}^{b^l} \quad (2.9)$$

Although aggregate endowment at time t , $Y_t^{b^l}$, is determined by the endowment path, population growth and time productivity paths, we need a closed form expression for aggregate consumption which can be obtained by solving the previous optimization

³see appendix A.

problem (2.1), with a *re-casted* overlapping multi-generations model in present value terms using the next proposition⁴:

Proposition 1 *The optimization problem given by equation(2.1), subject to (2.2),(2.3), and (2.4) is equivalent to maximize the same utility function with respect to consumption, subject to the equality of the present values of consumption \mathbb{C}_t and endowments \mathbb{Y}_t for T periods:*

$$\begin{aligned} \max_{\substack{c_{t+i}^{b^l+i} \\ c_{t+i}^{b^l+i}}} E_t \sum_{i=0}^{T-1} \beta^i U(c_{t+i}^{b^l+i}) \\ \text{s.t. } \mathbb{C}_t = \mathbb{Y}_t \end{aligned}$$

The solution expressions for consumption at every age are given by⁵:

$$\mathbb{E}_t c_{t+i+1}^{b^l+i+1} = \beta_{r_{t+i}} (1 + r_{t+i}) c_{t+i}^{b^l+i} \quad (2.10)$$

$$\text{with } c_t^{b^l} = \frac{\mathbb{Y}_t}{f(\beta_{r_t}, T)} \quad (2.11)$$

Expressions for the present value of consumption \mathbb{C}_t , and aggregate steady state consumption C , normalized to the size of the youngest generation b^l , are respectively given by:

$$\mathbb{C}_t = c_t^{b^l} f(\beta_{r_t}, T)$$

$$C = c^{b^l} f(\gamma_r, T)$$

where $\beta_{r_t} = \beta^{\frac{1}{\sigma}} (1+r_t)^{\frac{1-\sigma}{\sigma}} \stackrel{(\sigma=1)}{=} \beta$, $\gamma_r = \frac{\beta_r(1+r)}{(1+g)(1+z)}$, and $f(\beta_{r_t}, T) = \mathbb{E}_t \sum_{i=0}^{T-1} \prod_{k=0}^{i-1} \beta_{r_{t+k}}$.

Note that when the first argument is constant $f(\beta_r, T) = \sum_{i=0}^{T-1} \beta_r^i = \frac{1-\beta_r^T}{1-\beta_r}$.

Proof: The Euler equations of both problems are equivalent. The model is fully derived in appendix A. ■.

⁴from now on, and by default, when the subscripts and superscript are omitted in aggregates then we are referring to expressions normalized to the size of generation b^l at time t : $Y \equiv Y_t^{b^l}$.

⁵The model is deterministic.

Now that the model is solved for consumption, and that we derived a simple closed form expression for aggregate consumption in steady state, and by assuming aggregate endowment Y can also be represented by a closed form expression (as the endowment path is an exogenous set of parameters), then based on expression(2.9) we present in the next proposition a steady state closed form expression for excess borrowing, the *corner stone* to explicit the derivatives of equilibrium real interest rate with respect to age milestones:

Proposition 2 (i) *Excess borrowing in steady state is a continuous and differentiable function of the equilibrium real interest rate $r \in]-1, +\infty[$, and can be represented by the following expression:*

$$B(r, x) \begin{cases} \frac{1+r_{gz}}{r-r_{gz}}(Y - C) & \text{for } r \neq r_{gz} \\ -(1 + r_{gz})\frac{\partial C}{\partial r} & \text{for } r = r_{gz} \end{cases} \quad (2.12)$$

where $1 + r_{gz} = (1 + g)(1 + z)$, and x represents the exogenous parameters of the model.

(ii) $B(r, x) = 0$ has at least one solution if the no endowment retirement period duration $\equiv o$ is lower than the elasticity of inter-temporal substitution times the model duration, or $\frac{1}{\sigma} > \frac{o}{T-1}$, , and the duration of the initial no-endowment borrowing period $\equiv b$ is strictly lower than $T - 1$. Moreover, $\frac{\partial B}{\partial r}(r) < 0$ at least for one solution r solving $B(r, x) = 0$.

(iii) If $B(r, x) = 0$, the derivative of the real interest rate r with respect to any parameter of the model x can be expressed by:

$$\frac{\partial r}{\partial x} \equiv r_x = -\frac{B_x}{B_r} = -\frac{Y_x - C_x}{Y_r - C_r} = -\frac{\log_x Y - \log_x C}{\log_r Y - \log_r C} \quad (2.13)$$

Proof: (in appendix B) ■.

We have now the tools we need to algebraically formalize and interpret the ex-

pressions for the derivatives of steady state real interest rates with respect to age milestones at loan market equilibrium. This will be the purpose of the next section, for which we still need to formally express an endowment path, and respective aggregate and present value endowment expressions.

Note that for a constant endowment path for the total duration of the model, with no population, or productivity growth, and an elasticity of inter-temporal substitution equal to unity, then $1 + r = \frac{1}{\beta}$ solves $B(r, x) = 0$. This solution is equivalent to the equilibrium real interest rate in steady state of an infinitely lived single agent model, and assumes a uniform distribution of endowment through the duration of the model, with no retirement and no "*no endowment*" borrowing period b for the young households. From this *starting point* the introduction of a retirement period would correspond to a change of the income path that would reduce the steady state equilibrium real interest rate through an expansion of excess savings (equivalent to a contraction of excess borrowing). Moreover, the introduction of a no endowment period in the beginning of an agent's *economic life* would increase r , through an expansion of excess borrowing. Then changing the duration of relevant economic periods, through age-milestones changes, may affect households' income paths, and consequently excess borrowing through households' adjusted borrowing and saving needs in order to smooth households' life-time consumption paths. The resulting contraction or expansion of excess borrowing affects loan market equilibrium real interest rate.

Loan demand and supply

An equivalent way to understand the dynamics of excess borrowing in steady state is to split its expression B_t^{bl} into loan demand L_t^d and supply L_t^s . Loan demand for a given interest rate r is defined as excess borrowing of all households until the age they start to be savers, s^l . And loan supply is the negative expression of excess

borrowing of all households that are net savers, with age equal or above s^l :

$$L_t^d(r, v^i) = B_t^{d^b(r), b^l}(r, v^i) \text{ , where } d^b(r) = [b^l, s^l[\quad (2.14)$$

$$L_t^s(r, v^i) = -B_t^{d^s(r), b^l}(r, v^i) \text{ , where } d^s(r) = [s^l, d^L[\quad (2.15)$$

Note that $s^l \equiv s^l(r, v^i)$ is an endogenous variable of the model, which depends on r and age milestones v^i . Excess borrowing expressed in terms of loan demand and supply, is given by:

$$B_t^{b^l} = B_t^{d^b, b^l} + B_t^{d^s, b^l} = L_t^{d, b^l} - L_t^{s, b^l} \quad (2.16)$$

and loan market equilibrium can now be expressed by:

$$B_t^{b^l} = 0 \Leftrightarrow L_t^{d, b^l} = L_t^{s, b^l} \quad (2.17)$$

Where loan demand and supply may be expressed in terms of aggregate income and consumption during the respective periods⁶:

$$L_t^{d, b^l} = \frac{1 + r_{gz}}{r - r_{gz}} (Y^{d^b, b^l} - C^{d^b, b^l}) \quad (2.18)$$

$$L_t^{s, b^l} = -\frac{1 + r_{gz}}{r - r_{gz}} (Y^{d^s, b^l} - C^{d^s, b^l}) \quad (2.19)$$

This equivalent representation of excess borrowing can be helpful when interpreting how changes in age milestones affect equilibrium real interest rates, by comparing graphically steady state changes of loan demand and supply, as in Eggertsson and Mehrotra [21].

In the next section we analytically express and interpret how age milestone changes affect steady state equilibrium real interest rates.

⁶expressions derived in appendix

2.3 Real interest rate derivatives with respect to age parameters

In this section we present and interpret the derivatives of real interest rate with respect to age milestones $r_{v_i} \equiv \frac{\partial r}{\partial v_i}$. As age milestones are exogenous parameters of our model, closed form expressions for the derivatives of real interest rate with respect to age milestones are given directly using the tools described above, once we characterize households' endowment path with continuous and differentiable closed form expressions for present value of endowment \mathbb{Y} , and aggregate endowment Y .

We can interpret $r_{v_i} \equiv \frac{\partial r}{\partial v_i}$ as by how much r would have to change to compensate for the impact in excess borrowing of a change of a given age milestone v_i , so that loan market remains in equilibrium. By reasonably assuming that excess borrowing is a decreasing function of the real interest rate ⁷, or $B_r \equiv \frac{\partial B}{\partial r} < 0$, then an increase $dr > 0$ of the real interest rate would have a contraction impact on excess borrowing B , by $\frac{\partial B}{\partial r} dr < 0$. In order for loan market to remain in equilibrium B would have to increase back to 0, through a change ∂v_i of any given age milestone v_i , which should have an expansion effect on excess borrowing $\frac{\partial B}{\partial v_i} dv_i > 0$, such that:

$$dB(r, v_i) = \frac{\partial B}{\partial v_i} dv_i + \frac{\partial B}{\partial r} dr = 0 \quad (2.20)$$

Note that the expansion effect $\frac{\partial B}{\partial v_i} dv_i > 0$ implies that the change of age milestone ∂v_i has the same sign of the derivative of excess borrowing with respect to the age milestone, which means that if excess borrowing is a negative function of a given age milestone, then a decrease of this age milestone is required to compensate for an increase of the real interest rate, which is the same to say that r_{v_i} and B_{v_i} have the same sign, when $B_r < 0$. This is directly observed from the general expression

⁷From the proof of proposition 2 there are always more solutions for $B(r) = 0$ where $B_r < 0$ than otherwise. Although $B_r < 0$ for a wide calibration range, we could not find (yet) a sufficient condition for a unique solution where $B_r < 0$.

for r_{v_i} given by:

$$r_{v_i}(r, v) = \frac{dr}{dv^i} = -\frac{\frac{\partial B}{\partial v_i}}{\frac{\partial B}{\partial r}} = -\frac{B_{v_i}}{B_r}(r, v) \quad (2.21)$$

We also inspect how this relation is affected by other relevant parameters of the model like changes in time related productivity z_t , age related productivity ρ_t^i , and elasticity of inter-temporal substitution $\frac{1}{\sigma}$. With the same denominator B_r we expect that the derivatives of the real interest rate with respect to the exogenous parameters of the model keep a similar proportional relation. While B_v sets the sign of the real interest rate derivatives with respect to an age milestones, and the relative magnitude of the derivative with respect to others, B_r is the same denominator of those expressions setting a common amplitude factor. For example, for lower elasticities of inter-temporal substitution $\frac{1}{\sigma}$, aggregate consumption, and consequently excess borrowing, are expected to change less with real interest rate changes (B_r is flatter). Then, a stronger real interest rate reaction is required to compensate for the impact on excess borrowing of changing an age milestone, relative to a higher *EIS*. This phenomena can be observed in the calibration section.

B_{v_i} is the term that determines the relative signs and magnitudes of the derivatives with respect to each other, besides determining the sign of the derivative itself (given the sign of their common denominator B_r). Inspecting B_{v_i} is the purpose of the next proposition.

The missing pieces to derive a tractable closed-form expression for steady state excess borrowing are the closed-form expressions for present value and aggregate endowment. With a sufficiently generic endowment path, with no-endowment periods at the beginning and at the end of the model time span, with durations respectively given by $b \equiv d^b \geq 0$ and $o \equiv d^o \geq 0$, time and age type productivity growth rates such that $y_{t+1}^i = (1 + z)y_t^i$, and $y_t^{i+1} = (1 + \rho)y_t^i$, the endowment present value and

aggregate expressions in steady state are respectively given by:

$$\mathbb{Y}_t = \frac{y_t^{m^l}}{(1+r_z)^b} f\left(\frac{1+\rho}{1+r_z}, m\right) \quad (2.22)$$

$$Y_t = \frac{y_t^{m^l}}{(1+g)^b} f\left(\frac{1+\rho}{1+g}, m\right) \quad (2.23)$$

where $1+r_z = \frac{1+r}{1+z}$.

Proposition 3 *for $r > -1$ solving $B(r, v) = 0$, the partial derivatives of excess borrowing and equilibrium real interest rate with respect to age milestones and durations, can be expressed by,*

$$B_{v_i} = \frac{1+r_{gz}}{r-r_{gz}} Y (\text{Log}_{v_i} Y - \text{Log}_{v_i} C) \quad (2.24)$$

$$r_{v_i} = -\frac{B_{v_i}}{B_r} = \frac{Y_{v_i} - C_{v_i}}{C_r} = \frac{\text{Log}_{v_i} Y - \text{Log}_{v_i} C}{\text{Log}_r C} \quad (2.25)$$

(i) *The derivatives of the natural rate of interest with respect to age milestones can be expressed in terms of the derivatives of the natural rate of interest with respect to the durations of the young borrowing period b , the duration of middle age m , and the duration of the model T (here the duration of retirement o is a dependent variable):*

- *Adulthood:* $r_{b^l} = -r_b - r_T$
- *First job:* $r_{m^l} = r_b - r_m$
- *Retirement:* $r_{o^l} = r_m$
- *Life expectancy:* $r_L = r_T$

(ii) *for a sufficiently generic households' endowment path expressed by $y_t^i > 0$ for $i \in [m^l, m^h]$, and $y_t^i = 0$ for $i \in [b^l, m^l] \cup [m^h, d^L]$, where time and age related productivity growth rates, for $i \in [m^l, m^h]$, respectively given by $1+z_t^i = \frac{y_{t+1}^i}{y_t^i}$, and $1+\rho_t^i = \frac{y_{t+1}^{i+1}}{y_t^i}$, the partial derivatives B_{v_i} and r_{v_i} have the following signs:*

◆ *Partial derivatives with respect to durations:*

CHAPTER 2. AGE MILESTONES AND LOW INTEREST RATES

- Young borrower	$B_b > 0$	$\xRightarrow{B_r < 0}$	$r_b > 0$
- Middle age	$B_m > 0$	$\xRightarrow{B_r < 0}$	$r_m > 0$
- Model duration	$B_T < 0$	$\xRightarrow{B_r < 0}$	$r_T < 0$

◆ *Partial derivatives with respect to age milestones:*

- Adulthood:	$B_{b^l} < 0$	$\xRightarrow{B_r < 0}$	$r_{b^l} < 0$
- First job:	$B_{m^l} > 0$	$\xRightarrow{B_r < 0}$	$r_{m^l} > 0$
- Retirement:	$B_{o^l} > 0$	$\xRightarrow{B_r < 0}$	$r_{o^l} > 0$
- Life expectancy:	$B_{d^L} < 0$	$\xRightarrow{B_r < 0}$	$r_{d^L} < 0$

Proof: in appendix C. ■

As the relative signs of the partial derivatives of real interest rate are determined by the partial derivatives of excess borrowing with respect to age milestones and durations, given by (2.24), we next present those expressions for interpretation. The partial derivatives of excess borrowing w.r.t. periods duration d are given by:

$$B_b = Y\beta_r \Delta \log(\gamma_r, \beta_r) > 0 \quad (2.26)$$

$$B_m = Y \frac{1+\rho}{1+r_z} \Delta H^m \left(\frac{1+\rho}{1+g}, \frac{1+\rho}{1+r_z} \right) > 0 \quad (2.27)$$

$$B_T = -Y\beta_r \Delta H^T(\gamma_r, \beta_r) < 0 \quad (2.28)$$

and the partial derivatives of Excess Borrowing w.r.t. age milestones:

$$B_{b^l} = -Y\beta_r \Delta H^{-T}(\gamma_r, \beta_r) = -B_b - B_T < 0 \quad (2.29)$$

$$B_L = -Y\beta_r \Delta H^T(\gamma_r, \beta_r) = B_T < 0 \quad (2.30)$$

$$B_{m^l} = Y \frac{1+\rho}{1+r_z} \Delta H^{-m} \left(\frac{1+\rho}{1+g}, \frac{1+\rho}{1+r_z} \right) = B_b - B_m > 0 \quad (2.31)$$

$$B_{o^l} = Y \frac{1+\rho}{1+r_z} \Delta H^m \left(\frac{1+\rho}{1+g}, \frac{1+\rho}{1+r_z} \right) = B_m > 0 \quad (2.32)$$

where $\triangle f(x, y) = \frac{f(x)-f(y)}{x-y} > 0$ if $f' > 0$; $H^a(x) \equiv \frac{1}{a} \frac{\log x^{-a} - \log 1}{x^{-a} - 1}$, and $H^{a'} > 0$ ⁸.

All age-milestones expressions have a similar look. (i) The superscript parameter of function H is the duration the period affected by the change of the age milestone. The sign of the superscript corresponds to the change sign of the period duration with an increase of the age milestone. For example, if b^l increases then the duration of the model T will decrease, and the superscript $-T$ is used for the function H . (ii) We can note also that the ratio of any two arguments is the same, or $\frac{\gamma_r}{\beta_r} = \frac{\frac{1+\rho}{1+g}}{\frac{1+\rho}{1+r_z}} = \frac{1+r}{1+r_{gz}}$.

2.3.1 Durations of relevant lifetime economic periods

Model duration T : An increase of model duration T expands the expected retirement duration for the same amount, $o = T - m - b \Rightarrow \partial o = \partial T$, assuming retirement duration o is the dependent variable. This increases households savings needs during middle age for the same steady state real interest rate level, corresponding to a reduction of excess borrowing. Consequently the partial derivative of excess borrowing with respect to T is negative. For the loan market to remain in equilibrium a positive compensation of excess borrowing is required by an adjustment of the real interest rate, which must be negative if the partial derivative of excess borrowing with respect to the natural rate of interest is negative too, our base case by assumption. A positive change in life expectancy L , calling for a negative change in r leads to a negative $r_L = \frac{\partial r}{\partial L}$.

Endowment duration m : An increase of endowment duration m contracts the expected retirement duration for the same amount, $o = T - m - b \Rightarrow \partial o = -\partial m$. This reduces households savings needs during middle age, corresponding to an increase of excess borrowing. The mechanism is the opposite as the one described above for T .

⁸see proof of proposition 3.

Initial no-endowment duration b : An increase of b has a double positive effect on excess borrowing. The first by expanding initial borrowing period by ∂b with a positive effect on excess borrowing, and a second by contracting the retirement period by $\partial o = -\partial b$ which reduces household saving needs with a further positive impact on excess borrowing.

The partial derivatives of real interest rates and excess borrowing share the same sign, if $B_r < 0$, and are analytically expressed by:

$$r_b = \frac{\log \gamma_r - \log \beta_r}{\log_r C} > 0 \quad (2.33)$$

$$r_m = \frac{H^T \left(\frac{1+\rho}{1+g} \right) - H^T \left(\frac{1+\rho}{1+r_z} \right)}{\log_r C} > 0 \quad (2.34)$$

$$r_T = -\frac{H^T \gamma_r - H^T \beta_r}{\log_r C} < 0 \quad (2.35)$$

2.3.2 Age milestones bounding the duration of the model

Age milestones b^l and $b^L \equiv L$ limit the model duration $d^T \equiv T = L - b^l + 1$.

Age milestone L : When only life expectancy ⁹ increases among all age milestones, then the model duration T increases by the same amount, $T = L - b^l + 1 \Rightarrow \partial T = \partial L$, increasing the expected retirement duration too¹⁰, $o \equiv d^o = L - m^h + 1 \Rightarrow \partial o = \partial L = \partial T$, impacting negatively excess borrowing by the same mechanism described above for the model duration T .

Age milestone b^l : Furthermore, increasing the age of adulthood b^l shortens the model duration T by the same amount, as well as the duration of borrowing period $d^b = m^l - b^l$. This contraction of the initial borrowing period leads to a reduction of excess borrowing for the same real interest rate level.

$B_{b^l} = -B_b - B_T$: Note that an increase of adulthood age can be interpreted as a

⁹although we use the term *expectancy* the model is deterministic

¹⁰We are inspecting partial derivatives, which means that only one parameter changes.

combination of changes in two periods: a contraction of the young borrowing period b , combined with a contraction of the total duration of the model T when the retirement duration is a dependent variable. As we have seen above, a contraction of b alone as a double negative effect on excess borrowing. The first one is through a reduction of loans demand for the same interest rate level, and a second one is an increase of loans supply through an expansion of the retirement period, in order to keep the durations of middle age and model period, m and T , unchanged. The second effect, the retirement duration increase, is directly offset by the reduction of T which only impacts retirement duration. The combined effect is a contraction of the borrowing period b , leaving m and o unchanged. So $r_{b'}$ would be equal to $-r_b$ if the dependent duration parameter was T .

The partial derivatives of excess borrowing with respect to age milestones bounding the duration of the model T are both negative. The consequence is the same, although triggered by different mechanisms: (i) Increasing life expectancy that expands loan supply, and (ii) increasing the age of adulthood that contracts loan demand. The partial derivatives of real interest rates have the same negative sign, if $B_r < 0$, and are analytically expressed by:

$$r_{b'} = -\frac{H^{-T}(\gamma_r) - H^{-T}(\beta_r)}{\log_r C} = -r_b - r_T \quad (2.36)$$

$$r_L = -\frac{H^T(\gamma_r) - H^T(\beta_r)}{\log_r C} = r_T \quad (2.37)$$

2.3.3 Age milestones bounding labor income duration

Age milestones m^l and $o^L \equiv L$ limit the endowment duration $d^m \equiv m = o^l - m^l$.

Age milestone o^l : An increase of the retirement age o^l contracts expected retirement duration, reducing households' saving needs, equivalent to increasing excess borrowing for the same real interest rate level. The mechanism is the same as a reduction

of life expectancy described above.

Age milestone m^l : Furthermore, increasing the age of first job m^l expands the duration of the borrowing period $d^b = m^l - b^l$, which causes an increase of excess borrowing for the same real interest rate level.

$B_m^l = B_b - B_m$: Note that an increase of the age of first job can be interpreted as a combination of changes in two periods: an expansion of the young borrowing period b , combined with a contraction of the middle age duration m , when the retirement duration is a dependent variable. As we have seen above, an expansion of b alone as a double positive effect on excess borrowing. The first one is via an increase of loans demand for the same interest rate level, and a second one is via a reduction of loans supply through an contraction of the retirement period, in order to keep the durations of middle age and the model, m and T , unchanged. The second effect, the reduction of the retirement duration, is directly canceled by the reduction of the middle age period m which increases back the retirement duration by the same amount. The combined effect corresponds to an expansion of the borrowing period b that leaving retirement duration o unchanged. r_{m^l} would be equal to r_b if T was the dependent variable.

The partial derivatives of excess borrowing with respect to age milestones bounding endowment duration are both positive. Same consequence triggered by different mechanisms: Increasing retirement age that contracts loan supply, and increasing age of first job that expands loan demand. The partial derivatives of real interest rates have the same negative sign, if $B_r < 0$, and are analytically expressed by:

$$r_{m^l} = \frac{H^{-m} \left(\frac{1+\rho}{1+g} \right) - H^{-m} \left(\frac{1+\rho}{1+r_z} \right)}{\log_r C} = r_b - r_m \quad (2.38)$$

$$r_{o^l} = \frac{H^m \left(\frac{1+\rho}{1+g} \right) - H^m \left(\frac{1+\rho}{1+r_z} \right)}{\log_r C} = r_m \quad (2.39)$$

Equation (2.25) can also be used to formalize algebraically the partial derivatives of

the natural rate of interest with respect to other exogenous parameters of the model, namely the population growth rate g , total factor productivity growth z , and age dependent endowment growth rate ρ , whose relative signs are given by expression (2.24).

2.4 Intergeneration transfers

Until now we assumed that agents only interact with each-other by borrowing and lending. We now introduce transfers between generations in the form of bequests to children, gifts to parents, and a *pay-as-you-go* social security system, as those concepts are closely related to agents age structure. We will see how endogenous bequest and gifts are affected by changes in age structure, and how a social security tax may mitigate or amplify those changes.

2.4.1 Intergenerational Altruism

Imagine there are intergenerational altruistic linkages between parents and children, taking the form of transfers between agents during the last year of their lives and their direct descendants. We start by assuming that transfers are positive corresponding to positive bequests left by parents to their children, but we also analyze the case of children caring about their parents wealth. We start by recasting the budget constraints and derive a re-casted expression for excess borrowing as a function of the excess borrowing expression without intergenerational transfers, which will be valid for the two alternative preference functions used later in the sub-section. Then we endogenize the bequest motive by adjusting agents' preferences, to inspect how bequests and gifts are affected by changes of age milestones, and vice-versa. We use two modeling methods: first a *Warm glow* bequest motive type, where parents value the bequest itself. This bequest motive is the one we use to calibrate the model

in the last section. And second, an *altruism* bequest motive type, where parents value their children's utility.

(i) *Recasting budget constraints with bequests*

Let inter-generational altruism be represented by the bequest level Q_t^T left or received by an agent to or from his direct descendants during the last year of his live T at time t . A positive Q refers to *forward* altruism from parents to children, where $Q^f \equiv Q > 0$. And a negative Q refers to *backward* altruism from children to parents, where $Q^b \equiv -Q > 0$. Without loss of generality, and in order to keep the model tractable the same mechanism is used in both cases. Let $Q_t \equiv Q_t^T$, μ be the age difference between parents and children, and n the number of children per agent. Note that $n = (1 + g)^\mu$. The budget constraints have adjusted expressions when bequests are received at the age $b^l + T - 1 - \mu$, and at life expectancy $b^l + T - 1$:

$$c_{t+T-1-\mu}^{T-\mu} = y_{t+T-1-\mu}^{T-\mu} + b_{t+T-1-\mu}^{T-\mu} - (1 + r_{t+T-2-\mu})b_{t+T-2-\mu}^{T-1-\mu} + \frac{Q_{t+T-1-\mu}}{n} \quad (2.40)$$

$$c_{t+T-1}^T = y_{t+T-1}^T - (1 + r_{t+T-2})b_{t+T-2}^{T-1} - Q_{t+T-1} \quad (2.41)$$

Adjusted present value budget constraint and excess borrowing expressions in steady state are given by:

$$\mathbb{C}_t^{b^l} = \mathbb{Y}_t^{b^l} + \frac{r - r_{gz}}{1 + r_{gz}} M^q Q_t \quad (2.42)$$

$$B_t^q = B_t - Q_t G^q, \text{ where } G^q = M^q \frac{f(\gamma_r, T)}{f(\beta_r, T)} \quad (2.43)$$

$$\text{where } M_q = \frac{(1 + r_{gz})^{T-\mu}}{(1 + r)^{T-1}} \Delta x^\mu(1 + r, 1 + r_{gz}) \quad (2.44)$$

$$\text{and } \Delta x^\mu(y, z) = \frac{y^\mu - z^\mu}{y - z} > 0 \quad (2.45)$$

Because M_q is positive, G_q is also positive, from where it is straightforward to derive the impact of bequests on the natural rate of interest from a *no-bequest* initial state: Let r^q solve $B^q(r^q) = 0$ with an operative bequest motive, and Let r solve $B(r) = 0$

otherwise, with an inoperative bequest motive. Note that $B(r^q)$ has the same sign of Q because $B^q(r^q) = 0 \Leftrightarrow B(r^q) = QG^q(r^q)$ and $G^q(r^q) > 0$. Because we are assuming that $B(r)$ is a negative function of r we have:

$$Q > 0 \Rightarrow B(r^q) > 0 \Rightarrow r^q < r \quad (2.46)$$

$$Q < 0 \Rightarrow B(r^q) < 0 \Rightarrow r^q > r \quad (2.47)$$

From where bequest from parents to children reduce the equilibrium real interest rate from a *no-bequest* motive, and gifts from children to parents would have an opposite effect. By endogenizing the bequest motive we can derive Q , as well as all adjusted expressions of previous section.

(ii) *Warm glow bequest motive:*

To derive an expression for bequest Q using a *warm glow* motive, the previous utility function is adjusted according to the literature, by adding a bequest term to agent's utility function:

$$U_t = \max_{c_{t+i}^{1+i}} E_t \sum_{i=0}^{T-1} \beta^i u(c_{t+i}^{1+i}) + \beta^{T-1} \frac{u(Q_{t+T-1}^T)}{1 + \phi} \quad (2.48)$$

From FOC Q_{t+T-1} and c_{t+T-1} we get a steady state expression for bequests as a function of the present value of consumption $\mathbb{C}_t^{b^l}$:

$$Q_t = \frac{\mathbb{C}_t^{b^l}}{(1 + \phi)^{\frac{1}{\sigma}}} \frac{\gamma_r^{T-1}}{f(\beta_r, T)} > 0 \quad (2.49)$$

By combining the previous equation with the present value budget constraint (2.42) we obtain an expression for bequest Q_t given by:

$$Q_t = \frac{\mathbb{Y}_t}{(1 + \phi)^{\frac{1}{\sigma}} \frac{f(\beta_r, T)}{\gamma_r^{T-1}} - \frac{r - r_{gz}}{1 + r_{gz}} M_q} \quad (2.50)$$

Those new tools would be enough to derive new closed-form expressions for the

partial derivatives of the real interest rates with respect to age milestones. We approach quantitatively that topic in the last section of this chapter.

We next inspect an alternative bequest motive where agents consider the utility of their descendants in their preference function. In that case, equilibrium real interest rates are constant while the bequest motive is active. We deriving the impact of changing age milestones on the bequest level similarly to previous sections:

$$\frac{\partial Q}{\partial v_i} \equiv Q_{v_i} = -\frac{B_{vi}^q}{B_Q^q} = \frac{B_{vi}^q}{G^q} \quad (2.51)$$

Q_{v_i} would have the sign of B_{vi}^q , since $G^q > 0$. And if B_{vi}^q and B_{vi} have the same sign then the results for Q_v would be the same as the ones for r_v . For example, an increase of life expectancy would reduce bequests left to children, and an increase of the retirement age would have the opposite effect. Let's then briefly study the model:

(iii) *Altruism bequest motive:*

We now assume that bequests reflect agent's concern for the welfare of their descendants, by weighting children utility in agent's utility function, in the spirit of Barro [4]. By using the same mechanism we also examine agents concern with their parents, following the approach of Blanchard and Fischer [7]. The utility function takes the expression given below for both cases, where the utility of descendants is discounted in agent's utility with a lag of μ years (age difference between parents and children). The discount factor is β weighted by ϕ^f a selfish parameter (Barro and Sala-i Martin [5]) greater than one when parents prefer an additional unit of self consumption to a unit of children consumption in the same year. We assume that the utility of descendants in agents' preference function is not affected by the

number of children (Blanchard and Fischer [7]):

$$U_t^0 = \max_{c_{t+i}^{b^l+i}} E_t \sum_{i=0}^{T-1} \beta^i u(c_{t+i}^{b^l+i}) + \left(\frac{\beta}{1 + \phi^f} \right)^\mu U_{t+\mu}^1 = V_t^0 + \left(\frac{\beta}{1 + \phi^f} \right)^\mu U_{t+\mu}^1 \Leftrightarrow \quad (2.52)$$

Where $V_t^0 = \max_{c_{t+i}^{b^l+i}} E_t \sum_{i=0}^{T-1} \beta^i u(c_{t+i}^{b^l+i})$. We can solve (2.52) recursively forward, and get:

$$U_t^0 = \sum_{j=0}^{\infty} \left[\left(\frac{\beta}{1 + \phi^f} \right)^\mu \right]^j V_{t+j\mu}^j \quad (2.53)$$

The budgets constraints of this maximization problem are the same considered at the beginning of the current section, given by expressions (2.40)(2.41). With no restrictions on the sign and level of the intergenerational transfer from an agent to his descendants during the last year of his life, the equilibrium real interest rate would have the following expression in steady state:

$$1 + r^{\phi^f} = \frac{1 + \phi^f}{\beta} (1 + r_{gz}) \quad (2.54)$$

The bequest parameter Q^f is directly derived from loan market equilibrium expression given by equation (2.43):

$$B^q(r^{\phi^f}) = 0 \Leftrightarrow B(r^{\phi^f}) - Q^f G^q(r^{\phi^f}) = 0 \Rightarrow Q^f = \frac{B(r^{\phi^f})}{G^q(r^{\phi^f})} \quad (2.55)$$

We have seen that $G^q > 0$. Then a positive forward bequest Q^f implies that $B(r^{\phi^f}) > 0$, and consequently $r^{\phi^f} < r$, where r is the natural rate of interest of the maximization problem without bequest, since we are assuming that excess borrowing without bequest $B(r)$ is decreasing with r and $B(r) = 0$. Then, if $r^{\phi^f} < r$, the natural rate of interest with forward altruism r^f , and Q^f are given by:

$$r^{\phi^f} < r \Rightarrow r^f = r^{\phi^f}, \text{ and } Q^f = \frac{B(r^{\phi^f})}{G(r^{\phi^f})} > 0 \quad (2.56)$$

Otherwise, if $B(r^{\phi^f}) \leq 0 \Rightarrow r^{\phi^f} \geq r$, then there is no loan market equilibrium with a positive bequest. Consequently $Q^f = 0$, and the excess borrowing expression with a forward altruism bequest motive B^q is the same as in the problem with an inoperative bequest motive, and the same for the natural rate of interest:

$$r^{\phi^f} \geq r \Rightarrow r^f = r, \text{ and } Q^f = 0 \quad (2.57)$$

To summarize, parents leave bequests to their children only if their natural rate of interest without the possibility of bequests is greater than r^{ϕ^f} . Otherwise they will leave no bequests to future generations independently of their degree of altruism.

$$r^f = \min(r, r^{\phi^f}) \quad (2.58)$$

$$Q^f = \frac{B}{G^q}(r^f) \geq 0 \quad (2.59)$$

Note that the derivatives of bequest and excess borrowing with an inoperative bequest motive with respect to age milestones bounding the endowment period, Q_v and B_v , have the same sign¹¹. Then the changes in age milestones that affect the natural rate of interest without bequest will affect bequest levels in the same direction, when the motive is active. For example increasing retirement age will motivate parents to increase the bequest to their children. In the next section we confirm that the same result is robust also for age milestones bounding the duration of the model, L and b^l . Then an increase of life expectancy would reduce bequests left to children.

Backward Altruism: Agents concerned with their parents' wealth

We now assume instead that agents are concerned with their parents wealth, by helping them during the last year of their parents' lives. The budget constraints are the same but Q is now negative. The backward transfer parameter from children to

¹¹ G^q is constant with respect to m^l and o^l .

parents is given by $Q^b = -Q$, and the backwards altruistic parameter by ϕ^b . The mechanism is the same as before, with utility function represented by:

$$U_t^0 = V_t^0 + \left(\frac{\beta}{1 + \phi^b} \right)^{-\mu} U_{t-\mu}^{-1} \Leftrightarrow \quad (2.60)$$

$$U_t^0 = \sum_{j=0}^{-\infty} \left[\left(\frac{\beta}{1 + \phi^b} \right)^{\mu} \right]^j V_{t+j\mu}^j \quad (2.61)$$

The resulting expressions for the natural rate of interest and transfers from agents to their parents are a mirror of the above:

$$r^b = \max(r, r^{\phi^b}) \quad (2.62)$$

$$Q^b = -\frac{B}{G^q}(r^b) = -Q \geq 0 \quad (2.63)$$

where,

$$1 + r^{\phi^b} = \frac{1 + \phi^b}{\beta}(1 + r_{gz}) \quad (2.64)$$

Agents help their parents only if their natural rate of interest with an inoperative bequests motive is lower than r^{ϕ^b} . Furthermore, and because the derivatives of Q^b and excess borrowing without bequest $B(r)$ with respect to age milestones have opposite signs, the changes in age milestones that affect the natural rate of interest with an inoperative bequest motive will affect bequest levels in the opposite direction too. For example increasing retirement age will motivate agents to be less generous with their parents, and a longer life will have the opposite effect.

Two Sided Altruism: Agents concerned with children and parents wealth

We now combine the motivations above by assuming that agents have the choice of helping their children or their parents. This assumption simplifies the maximization problem as the same budget constraints can be used, where Q can be positive or negative, but not both at the same time. We further assume that agents will prior-

itize supporting their parents at the end of their lives, to future bequests for their children:

$$\phi^b \leq \phi^f \Leftrightarrow r^{\phi^b} \leq r^{\phi^f} \quad (2.65)$$

Combining the previous results for one sided forward and backward altruism we find the following expression for the natural rate of interest r^ϕ :

$$r^\phi = \begin{cases} r^{\phi^b} & \text{for } r < r^{\phi^b} & \Rightarrow Q^b > 0, Q^f = 0 \\ r & \text{for } r \in [r^{\phi^b}; r^{\phi^f}] & \Rightarrow Q^b = 0, Q^f = 0 \\ r^{\phi^f} & \text{for } r > r^{\phi^f} & \Rightarrow Q^b = 0, Q^f > 0 \end{cases} \quad (2.66)$$

When the natural rate of interest with an inoperative bequest motive changes significantly agents may change their altruistic behavior between their children and their parents. For example, a significant increase in life expectancy could change the motivation of agents from leaving a bequest to their children to helping their parents. Or, the increase of retirement age could have the opposite effect: parents would need less help, and more wealth would be available to help children. Furthermore, factors that contribute to lower the natural rate of interest without intergenerational altruism, like the increase of life expectancy, will decrease the propensity to leave a bequest to the next generation, and increase the willingness to help the previous one. Moreover in this model the natural rate of interest with and inoperative bequest motive increases with productivity. Assuming wealthier societies and agents are more productive, then the poorer would be more inclined to help their parents, and the richer their children.

We next introduce in our model a *social security tax*, to analyze how the equilibrium real interest rate, as well as bequests and gifts, are affected in the presence of social security transfers.

2.4.2 Social Security

We now introduce a *pay-as-you-go* social security system, where agents pay a tax τ on their income/endowment while they are working, and receive a pension after retirement equal to total collected social security contributions divided by the number of retired agents in each year. The budget constraints are now given by:

$$i \in [m^l, m^h] : c_{t+i-1}^i = (1 - \tau)y_{t+i-1}^i + b_{t+i-1}^i - (1 + r_{i-1})b_{t+i-2}^{i-1} \quad (2.67)$$

$$i \in [o^l, T-1] : c_{t+i-1}^i = b_{t+i-1}^i - (1 + r_{i-1})b_{t+i-2}^{i-1} + \tau \frac{Y_{t+i-1}}{N_{t+i-1}^o} \quad (2.68)$$

$$i = T : c_{t+T-1}^T = -(1 + r_{T-1})b_{t+T-1}^{T-1} + \tau \frac{Y_{t+T-1}}{N_{t+T-1}^o} \quad (2.69)$$

The present value budget constraint in steady state is now expressed by:

$$\mathbb{C}_t^{b^l, L} = \mathbb{Y}_t^{b^l, L} (1 - \tau M_\tau) \quad (2.70)$$

$$\text{where } M_\tau = 1 - \frac{Y_t^{b^l}}{\mathbb{Y}_t^{b^l, L}} \left(\frac{1+g}{1+r_z} \right)^{m+b} \frac{f\left(\frac{1}{1+r_z}, o\right)}{f\left(\frac{1}{1+g}, o\right)} \quad (2.71)$$

$$\text{and } \frac{Y_t^{b^l}}{\mathbb{Y}_t^{b^l, L}} = \left(\frac{1+g}{1+r_z} \right)^{-b} \frac{f\left(\frac{1+\rho}{1+g}, m\right)}{f\left(\frac{1+\rho}{1+r_z}, m\right)} \quad (2.72)$$

Using expression (2.70) in (2.11), we derive excess borrowing with social security based on (2.12), as the sum of the corresponding expression without social security with a positive term:

$$B^\tau(r, v, \tau) = B(r, v) + \tau G^\tau(r, v) \quad (2.73)$$

where $G^\tau(r, v)$ is positive¹²:

$$G^\tau(r, v) = \left(\frac{1+g}{r_z - g} \right) \mathbb{Y}_t^{b^l, L} M_\tau \frac{f(\gamma, T)}{f(\beta_r, T)} \quad (2.74)$$

The introduction of a social security system of this type corresponds to a forced redistribution of wealth from workers to the old. The need to save for retirement is expected to decrease. This fact is reflected by the positive term $\tau G^\tau(r, x)$ that expands excess borrowing, causing the steady state natural rate of interest r_n^τ to be greater than the one with no social security, r_n :

$$B^\tau(r_n^\tau, v, \tau) = 0 \Leftrightarrow B(r_n^\tau, v) = -\tau G^\tau(r, v) < 0 \Rightarrow r_n < r_n^\tau \quad (2.75)$$

or directly from (2.73), the derivative of excess borrowing B^τ with respect to τ is positive:

$$B^\tau(r, v, \tau) = \frac{\partial B^\tau}{\partial \tau}(r, v, \tau) = G^\tau(r, v) > 0 \quad (2.76)$$

The derivative of the natural rate of interest with respect to the social security tax given is positive¹³ and is expressed by:

$$r_\tau(r_n^\tau, v, \tau) \equiv \frac{dr}{d\tau}(r_n^\tau, v) = -\frac{G^\tau(r_n^\tau, v)}{B_r(r_n^\tau, v) + \tau G_r^\tau(r_n^\tau, v)} > 0 \quad (2.77)$$

Social Security and inter-generations Altruism

We now inspect how intergenerational altruism is affected by a social security tax, an exogenous compulsory transfer from younger to older generations. If agents care for their parents wealth without any social security system in place it is expectable that

¹²Note that $\frac{M_\tau}{r_z - g} > 0 \Leftrightarrow \frac{h_3(1+r_z) - h_3(1+g)}{r_z - g} > 0$ is true, because $h_3(x) = x \frac{f(\frac{x}{1+\rho}, m)}{f(\frac{1}{x}, o)}$ increases with x .

¹³We continue to assume that excess borrowing in the presence of this social security system has a negative slope with respect to the real interest rate.

any social security income would reduce the perceived level of help parents would need when old. Furthermore, if a social security system is sufficiently generous it is also expectable that elder agents become more motivated to help their children.

Let's first look to how a *pay-as-you-go* social security may affect backward altruism from agents to parents. The optimization problem is now given by agent's maximizing inter-generations altruism utility expression (2.52) subject to the budget constraints resulting from the direct combination of expressions (2.2),(2.3),(2.40),(2.41), with (2.67),(2.68),(2.69). The new excess borrowing expression that takes into account bequests and social security is an intuitive combination of the previous expressions, such that:

$$B^{\tau,b}(r^{\phi^b}, v, Q^b, \tau) = B(r^{\phi^b}, v) + \tau G^{\tau}(r^{\phi^b}, v) + Q^b G(r^{\phi^b}, v) = 0 \quad (2.78)$$

$$\text{and consequently, } Q^b = -\frac{B(r^{\phi^b}, v) + \tau G^{\tau}(r^{\phi^b}, v)}{G(r^{\phi^b}, v)} = \frac{B^{\tau}(r^{\phi^b}, v, \tau)}{G(r^{\phi^b}, v)} \geq 0. \quad (2.79)$$

Where Q^b is positive by assumption. Consequently,

$$Q^b > 0 \Leftrightarrow B^{\tau}(r^{\phi^b}, v, \tau) < 0 \Leftrightarrow \begin{cases} r_n^{\tau} < r^{\phi^b} \\ \tau < \tau^b = -\frac{B(r^{\phi^b}, v)}{G^{\tau}(r^{\phi^b}, v)} \Rightarrow B(r^{\phi^b}, v) < 0 \Rightarrow r_n < r^{\phi^b} \end{cases} \quad (2.80)$$

Children will care for their parents wealth when the social security tax is lower than the threshold τ^b . In that case the natural rate of interest is equal to r^{ϕ^b} , and the gift parameter Q^b from children to parents will change with the social security tax according to:

$$\frac{dQ^b}{d\tau} \equiv q_{\tau}^b = -\frac{G^{\tau}(r^{\phi^b}, v)}{G(r^{\phi^b}, v)} < 0 \quad (2.81)$$

A higher social security tax discourages transfers from children to parents, until a point when it starts encouraging bequests from parents to children. This is the

mechanism we analyze next.

If agents care for their children wealth without any social security system in place it is expectable that any social security income further increases the propensity to help the next generation. The optimization problem is the same as before, and forward altruistic transfers from children to parents require that:

$$B^{\tau,f}(r^{\phi^f}, v, Q^f, \tau) = B(r^{\phi^f}, v) + \tau G^{\tau}(r^{\phi^f}, v) - Q^f G(r^{\phi^f}, v) = 0 \quad (2.82)$$

$$\text{and } Q^f = \frac{B(r^{\phi^f}, v) + \tau G^{\tau}(r^{\phi^f}, x)}{G(r^{\phi^f}, v)} = \frac{B^{\tau}(r^{\phi^f}, v, \tau)}{G(r^{\phi^f}, v)} \geq 0 \quad (2.83)$$

Consequently,

$$Q^f > 0 \Leftrightarrow B^{\tau}(r^{\phi^f}, v, \tau) > 0 \Leftrightarrow \begin{cases} r^{\phi^f} < r_n^{\tau} \\ \tau > \tau^f = -\frac{B(r^{\phi^f}, v)}{G^{\tau}(r^{\phi^f}, v)} \Leftarrow B(r^{\phi^f}, x) > 0 \Leftarrow r^{\phi^f} < r_n \end{cases} \quad (2.84)$$

Parents will care for their children wealth when the social security tax is above a threshold τ^f . In that case bequest Q^f from agents to their children will change with the social security tax according to:

$$\frac{dQ^f}{d\tau} = q_{\tau}^f = \frac{G^{\tau}(r^{\phi^f}, x)}{G(r^{\phi^f}, x)} > 0 \quad (2.85)$$

An increase of the social security tax encourages forward altruism, which is also an intuitive result.

2.5 Quantitative calibration

In the previous sections we formalized algebraically the relation of changes of equilibrium real interest rates and evolving age milestones. Now we explain how those

milestone changes may account for around one third of the real interest rates reduction in US in recent years, by calibrating our model with capital during the period between 1985 and 2005.

We start by formally deriving the model with capital. Then we parametrized it to match some initial conditions in the initial steady state, namely the real interest rate, the capital output ratio $\frac{K}{Y}$, and the bequests to output ratio, using explicit values for age milestones in 1985. We use this calibrated version to calculate the derivatives of the real interest rate with respect to each age milestone $\frac{\partial r}{\partial v_i}$, and check how those values vary with changes of other parameters and variables of the model. Finally we estimate how much of the real interest rate reduction between 1985 and 2005 is explained by the age milestone changes observed during the period between 1985 and 2005 using our model.

2.5.1 OLG model with endogenous output and capital

Formally the model with endogenous output and capital is derived in a similar way¹⁴. Although the derivatives of the real interest rate with respect to age milestones can be easily derived, we do not give in this section their algebraic representation.

The household maximization problem, without bequests an social security for now, has the same appearance has before, and is given by:

$$\max_{c_{t+i}^{b^l+i}} E_t \left\{ \sum_{i=0}^{T-1} \beta^i U(c_{t+i}^{b^l+i}) \right\} \quad (2.86)$$

$$\text{s.t. } c_t^{b^l} = w_t^{b^l} l_t^{b^l} - a_t^{b^l} \quad (2.87)$$

$$c_{t+i}^{b^l+i} = w_{t+i}^{b^l+i} l_{t+i}^{b^l+i} + (1 + r_{t+i-1}) a_{t+i-1}^{b^l+i-1} - a_{t+i}^{b^l+i} \quad (2.88)$$

$$c_{t+T-1}^{o^h} = w_{t+T-1}^{o^h} l_{t+T-1}^{o^h} + (1 + r_{t+T-2}) a_{t+T-2}^{o^h-1} \quad (2.89)$$

¹⁴Appendix D.

where the same utility function is used, $U(c) = \frac{C^{\sigma-1}}{1-\sigma}$. Assets $a_t = k_t - b_t$ are composed by capital k_t that households rent to firms, and loans to other households $-b_t$. The capital portion of assets is always positive but the loans to other households $-b_t$ can be positive or negative. In any case, agents are called borrowers when $a_t < 0$ and savers otherwise. While employed each household is given an exogenous annual labor endowment that is assumed to increase with work experience at a constant rate¹⁵. The expression for labor endowment at age i is given by:

$$l_t^{m^l+i} = l_t^{m^l} (1 + \rho)^i \quad (2.90)$$

Without loss of generality we assume that the labor endowment in the beginning of the working period $l^{m^l} = 1$. Regarding households' asset composition $a_t = k_t - b_t$, we can rewrite the budget constraints in terms of loans and capital:

$$c_{t+i}^{b^l+i} = w_{t+i}^{b^l+i} l_{t+i}^{b^l+i} + \left[(1 - \delta) k_{t+i-1}^{b^l+i-1} + (1 + r_{t+i-1}^k) k_{t+i-1}^{b^l+i-1} - (1 + r_{t+i-1}) b_{t+i-1}^{b^l+i-1} \right] - \left[k_{t+i}^{b^l+i} - b_{t+i}^{b^l+i} \right] \quad (2.91)$$

where from the first order conditions of k_t and b_t , the *No-Arbitrage Condition (NAC)* is given by:

$$r_t^k = r_t + \delta \quad (2.92)$$

Similarly to the previous sections, solving the previous household optimization problem for consumption, without the bequests and social security modules, is equivalent

¹⁵We use this assumption to ensure algebraic tractability.

to solve:

$$\max_{c_{t+i}^{b^l+i}} E_t \left\{ \sum_{i=0}^{T-1} \beta^i U(c_{t+i}^{b^l+i}) \right\} \quad (2.93)$$

$$\text{s.t. } \mathbb{C}_t^{b^l} = \mathbb{W}_t^{b^l} \quad (2.94)$$

where $\mathbb{W}_t^{b^l}$ is the present value of wages instead of endowments. By introducing bequests and/or social security in the model, the present value budget constraint becomes:

$$\mathbb{C}_t^{b^l} = \mathbb{W}_t^{b^l} (1 + M_Q - \tau M_\tau) \quad (2.95)$$

where M_Q and M_τ are the same as derived in the previous section, with expressions respectively given by equations (2.44) and (2.72). The loan market equilibrium condition is still given by excess borrowing $B_t^{b^l} = 0$, which in steady state has the following representation:

$$B_t^{b^l} = \begin{cases} \frac{1+r_{gz}}{r-r_{gz}} (W_t^{b^l} - C_t^{b^l}) + K_t^{b^l} & \text{for } r \neq r_{gz} \\ -(1+r_{gz}) w_t \frac{\partial C_t^{b^l}}{\partial r} + K_t^{b^l} & \text{for } r = r_{gz} \end{cases} \quad (2.96)$$

Where $1 + r_{gz} = (1 + g)(1 + z_\alpha)$, and $1 + z_\alpha = (1 + z)^\frac{1}{\alpha}$. B^{b^l} is continuous and differential function for $r \in]-\delta, +\infty[$, from where the derivatives of the equilibrium real interest rate with respect to age milestones are also given by:

$$r_{v_i} \equiv \frac{\partial r}{\partial v_i} = - \frac{\frac{\partial B_t^{b^l}}{\partial v_i}}{\frac{\partial B_t^{b^l}}{\partial r}} \quad (2.97)$$

The model with capital with and without bequests and social security, is fully derived in appendix.

Table 2.1: Initial steady state parameters for 1985

Description	Parameter	Value
Emancipation	b^l	21.0
First job	m^l	23.0
Retirement	o^l	65.8
Life expectancy	d^L	74.6
Discount rate	β	0.978
Population growth	g	0.9%
TFP growth rate	z	0.0%
Age related productivity growth rate	ρ	0.0%
Intertemporal substitution	$\frac{1}{\sigma}$	0.5
Labor share	α	0.6
Depreciation rate	δ	0.1
Bequest parameter (warm glow)	ϕ	-0.6
Age children born	μ	25
Social security tax	τ	12.4%
Initial real interest rate	r	4.4%
Estate size/Output	Q/Y	2.2%
Capital to output ratio	K/Y	2.78

2.5.2 Quantifying derivatives of r with respect to age milestones

We start by parameterizing the initial steady state of the model with capital and bequests, as in table(3.3) for US in 1985. We use values for average life expectancy and retirement age from Knoema¹⁶. We ensure that the capital to output ratio $\frac{K}{Y} = \frac{1-\alpha}{r+\delta}$ is approximately¹⁷ 3. We use the bequest parameter ϕ to match the ratio $\frac{Q}{Y}$ around¹⁸ 0.02. We do not consider a social security tax in the base case scenario of the model. We derive the value of β by solving $B_t^{bl}(r = 4.4\%, v_i) = 0$, where 4.4% is the real interest rate corresponding to the initial steady state in 1985.

We then use the above parameters to calculate the derivatives of the real interest rate with respect to each age milestone using the expressions derived in previous sections,

¹⁶We assume that the ages of adulthood and first job are respectively equal to 21 and 23. We could alternatively have assumed, with similar results, that the age of adulthood was 18 instead, and that those two initial age milestones coincide, which might change the calibration of β .

¹⁷Brinca et al. [8].

¹⁸Hendricks [28]

Table 2.2: Derivatives with same parametrization, $\beta = 0.978$: changing r

Scenario	Derivatives of r with respect to age				r
	b^l	m^l	o^l	L	
E \equiv Endowment economy	-0.85%	0.70%	0.64%	-0.48%	-0.93%
E+Q \equiv Bequests	-0.89%	0.68%	0.68%	-0.48%	-1.62%
E+Q+ τ \equiv Social security tax	-0.64%	0.58%	0.38%	-0.29%	2.34%
K	-0.62%	0.72%	0.30%	-0.35%	4.83%
Base case: K+Q	-0.62%	0.69%	0.31%	-0.33%	4.40%
K+Q+ τ	-0.59%	0.69%	0.26%	-0.31%	6.42%

in several scenarios starting with an endowment economy and successively adding bequests, capital, and social security¹⁹. In table(2.2) we use the same parameters of our base case scenario, with endogenous output, capital, and a bequest motive, to calculate the derivatives of real interest rates with respect to age milestones, with and without capital, bequest and social security. Since all parameters besides ϕ , α , and τ are unchanged for all scenarios, the *natural rates of interest* r changes accordingly: r is higher when capital is introduced, lower when bequest are operative, and higher when social security is considered. Furthermore, we observe the following:

- The model can generate negative steady state equilibrium real interest rates, as can be observed in the Endowment Economy scenario of table(2.2).
- The signs of derivatives of real interest rates with respect to age milestones, r_{v_i} , are the ones expected: increasing life expectancy, and increasing age of adulthood have a negative impact on the real interest rate, respectively because savings expand, and borrowing contracts. Increasing the retirement age, as well as the age of first job, both impact positively the real interest rate, respectively because savings contract, and borrowing expands.
- When the capital weight in the model increases, an increase of the real interest rate ∂r has a greater expansion impact on loan supply, $S = -B$, to compen-

¹⁹Derivatives are calculated using each corresponding expressions for endowment economy scenarios, and compared with numerical estimations for all scenarios.

Table 2.3: Derivatives setting $r = 4.40\%$, adjusting β for equilibrium

Scenario	Derivatives of r with respect to age				β
	b^l	m^l	o^l	L	
E \equiv Endowment economy	-1.06%	0.99%	0.44%	-0.32%	0.944
E+Q \equiv Bequests	-1.10%	1.00%	0.44%	-0.30%	0.942
E+Q+ τ \equiv Social security tax	-0.75%	0.73%	0.35%	-0.27%	0.962
K	-0.58%	0.68%	0.31%	-0.36%	0.981
Base case: K+Q	-0.62%	0.69%	0.31%	-0.33%	0.978
K+Q+ τ	-0.45%	0.54%	0.26%	-0.31%	0.996

sate for the contraction of capital, requiring bigger changes in age milestones to sustain loan market in equilibrium. This reduces, in absolute terms, the derivatives of real interest rates with respect to age milestones, when capital is introduced in the model.

In table(2.3) we calibrate all the scenarios to match the steady state natural rate of interest of the base case, by adjusting β . We observe that the derivatives corresponding to higher age milestones (retirement age, and life expectancy) become more similar across scenarios, and that in general the orders of magnitude do not change significantly.

We continue to test the robustness of the model in table (2.4), where we can also observe that the derivatives of the real interest rates with respect to age milestones, are generally less sensible to changes of the main parameters of the model, with the exception of the elasticity of inter-temporal substitution $\frac{1}{\sigma}$.

As we can observe in table(2.5), the constant relative risk aversion coefficient σ is a determinant factor driving the magnitude of the derivatives. Excess borrowing is more rigid for higher levels of σ , calling for a lower change of an age milestone in absolute terms, to compensate the same real interest rate change, in order to keep loan market in equilibrium. Consequently the derivative corresponding to a greater σ will be greater. In table(2.5) we observe a relation of quasi proportionality between real interest rate derivatives and σ : The elasticity of intertemporal substitution then

Table 2.4: Robustness Analysis

parameter $\equiv x$	∂x	$y = r_{v_i}(x + \delta x)/r_{v_i}(x)$			
		y_{b^l}	y_{m^l}	y_{o^l}	y_L
$\frac{1}{\sigma}$	$0.5 \rightarrow 1.0$	0.4	0.4	0.5	0.5
g	$0.9\% \rightarrow 0.0\%$	0.9	0.9	1.1	1.1
z	$0.0\% \rightarrow 1.0\%$	1.3	1.3	1.0	1.0
ρ	$0.0\% \rightarrow 1.0\%$	1.1	1.0	1.1	1.1
δ	$0.1 \rightarrow 0.2$	1.2	1.1	1.4	1.2
α	$0.6\% \rightarrow 0.7\%$	1.0	1.0	1.2	1.0
ϕ	$-0.6 \rightarrow 0.0$	1.0	1.0	1.0	1.0
τ	$0.0\% \rightarrow 12.4\%$	1.0	1.0	0.8	0.9

Table 2.5: Relative Risk Aversion

σ	Derivatives of r with respect to age			
	b^l	m^l	o^l	L
0.5	-0.14%	0.15%	0.07%	-0.07%
1	-0.30%	0.31%	0.14%	-0.15%
2	-0.62%	0.69%	0.31%	-0.33%
3	-0.96%	1.19%	0.50%	-0.54%
4	-1.32%	1.91%	0.74%	-0.79%

Table 2.6: Simulation Results

Scenario	Age milestones				natural rate of interest: r			
	b^l	m^l	o^l	L	$\sigma = 1$	$\sigma^* = 2$	$\sigma = 3$	$\sigma = 4$
Base case: 1985	21.0	23.0	65.8	74.6	4.4%	4.4%	4.4%	4.4%
2005	21.0	23.0	64.6	77.5	3.7%	2.5%	1.2%	-0.3%
2015	23.0	25.0	64.6	79.0	3.5%	2.2%	0.9%	-0.6%
$\Delta(1985, 2005)$	0.0	0.0	-1.2	+2.9	-0.7%	-1.9%	-3.2%	-4.7%
$\Delta(1985, 2015)$	+2.0	+2.0	-1.2	+4.4	-0.9%	-2.2%	-3.5%	-5.0%

$\sigma^* = 2$ is the base case scenario.

seems a decisive factor when age structure is considered, as it can greatly influence the conclusions and quantitative results from a calibrated model. This becomes clear in table (2.6) where, in our base case scenario with $\sigma = 2$, changes in age milestones could explain around half of the real interest rate decline from 1985 to 2005/2015 in US, considering a real interest rate equal to 4.4% in 1985 and around zero in 2015²⁰. We use observed values for average retirement age and life expectancy, as well population annual growth rate²¹. But if we calibrate the model using $\sigma = 1$, the same age milestone changes would account for around one fourth of the total real interest rate decline during the period in study. And if σ takes the value of 4 then the demographic changes during the same period would be able to explain the full real interest rate decline.

We can conclude with the observation that the constant relative risk aversion coefficient may work as an amplification factor of the impact of demographic changes on the *natural rate of interest*.

²⁰We assume that young agents postpone by two years their adulthood, first job, and first child in 2015, relativity to 2005.

²¹The reduction of population annual growth rate from 0.9% in 1985 to 0.7% in 2005 explains a reduction of the real interest rate of around -0.2%.

2.6 Final remarks

In this chapter we formalized the relation between real interest rates and relevant age milestones of a person's life, using an overlapping multi-generations model where one generation correspond to one year.

Although the impact of age structure in relevant World economies has been a recurrent topic covered in recent literature, in particular to explain the persistent decline of interest rates and economic stagnation, there has not yet been an attempt, to the best of our knowledge, to formally derive the analytic relations between real interest rates and each age milestone. The purpose of this chapter is to fill out this gap, by deriving tractable algebraic real interest rate elasticity expressions with respect to each age parameter, in order to formalize the mechanisms by which real interest rate changes occur. This allows, for example, to algebraically derive precisely by how much the natural rate of interest may be permanently dragged down eventually to negative levels, by any combination of increasing life expectancy, postponing first child birth, lowering the retirement age, increasing adulthood age, or reducing the age of first job. Moreover, we can use our framework to quantify the contribution of population age structure changes, to the decline of real interest rates in recent years, which can be close to one half according to a calibrated version of our age milestones toolbox.

The main underlying mechanism relating age milestones and real interest rates in our model relies on the relative place and duration of labor income with respect to life expectancy when agents smooth consumption. For example, a longer retirement period, resulting from a reduction of the retirement age or an increase of life expectancy, makes households save more, thus expanding the supply of loans which drags down the real interest rate that ensures equilibrium in the loan market. We also inspect how the exogenous parameters of the model, namely the elasticity of inter-temporal substitution, productivity growth, income growth path of house-

holds, and population growth, may amplify or mitigate the impact of age milestones changes on the natural rate of interest. In addition we inspected how inter-generation transfers are affected by age milestone changes. For example, why and how increasing life expectancy may reduce endogenous bequest levels, decrease the propensity to help children, or increase the willingness to help parents.

Laterally to our main contribution in this chapter, the analytic formulation of interest rate changes with respect to age milestones, we also developed a tractable algebraic framework to solve overlapping multi-generations optimization problems with a demographic structure.

2.A Proposition 1: Present Value and Aggregate Consumption

By expressing b_{t+T-2}^{T-1} in the third budget constraint (2.4) in terms of endowment and consumption, and recursively substituting the expressions for b_{t+i}^i we can derive the following equality between present values of expected households' consumption and income paths, beginning adulthood at time t :

$$\mathbb{C}_t^{b^l, d^L} = \mathbb{Y}_t^{b^l, d^L} \quad (2.A.1)$$

where $\mathbb{C}_t^{l,h}$ is the present value of expected future and present consumption of an agent with age l at time t , until age h at time $t+(l-h)$ - and the same for endowment - given by expressions:

$$X_t^{l,h} = \mathbb{E}_t \sum_{i=0}^{h-l} \frac{x_{t+i}^{l+i}}{(1+r)_t^i}, \text{ where } (1+r)_t^i = \prod_{j=0}^{i-1} (1+r_{t+j}) \quad (2.A.2)$$

We omit the present value upper bound when it is equal to d^L , or $\mathbb{X}_t^{l,h} \equiv \mathbb{X}_t^l$. Equation (2.A.1) is intuitive, and corresponds to a *present value* budget constraint derived from the previous ones, (2.2) to (2.4). It means that the present value of lifetime consumption of an household is equal to the present value of its endowment path. The borrowing/saving path of an household, or its loan path, is an enabler of its optimal consumption path, given its income path.

A more general version of the present value budget constraint, at any given age, is given by the following expression:

$$\mathbb{C}_t^{b^l+i} = \mathbb{Y}_t^{b^l+i} - (1+r_{t-1})b_{t-1}^{b^l+i-1} \quad (2.A.3)$$

This expression is also self-explanatory: The forward present value of consumption at time t is equal to the present value of the endowment path minus debt and

interest costs from previous period (or plus savings and interest income from previous period). Equations (2.A.1) and (2.A.3) will be frequently used, in particular to derive expressions for equilibrium consumption, for which we previously derive from consumption *First Order Conditions* the consumption general Euler equation for this model, given by:

$$\mathbb{E}_t \frac{c_{t+1}^{b'+i+1}}{1+r_t} = \beta_{r_t} c_t^{b'+i} \quad (2.A.4)$$

where,

$$\beta_{r_t} = \beta^{\frac{1}{\sigma}} (1+r_t)^{\frac{1-\sigma}{\sigma}} \stackrel{(\sigma=1)}{=} \beta \quad (2.A.5)$$

By using the Euler equation (3.A.13) in the present value general expression given by (2.A.2), we get an expression for the present value of consumption dependent on household's effective consumption at time t and expected real interest path, given by:

$$\mathbb{C}_t^{b'+i} = c_t^{b'+i} \sum_{j=0}^{T-i-1} \prod_{k=0}^{j-1} \beta_{r_{t+k}} = c_t^{b'+i} f(\beta_{r_t}, T-i) \quad (2.A.6)$$

where $f(x_t, n) = \sum_{j=0}^{n-1} \prod_{k=0}^{j-1} x_{t+k}$. Note that when x is constant $f(x, n) = \sum_{j=0}^{n-1} x^j = \frac{1-x^n}{1-x}$. This simplifies the algebra in the special case of a log utility function for $\sigma = 1$ and $\beta_r = \beta$, as well as when formulating expressions for steady state. Combining the previous equation (2.A.6) at the age of adulthood with the present value budget constraint given by (2.A.1), we can express consumption at the age of adulthood in terms of expected real interest rate and income paths:

$$c_t^{b'} = \frac{\mathbb{Y}_t^{b'}}{f(\beta_{r_t}, T)} \quad (2.A.7)$$

A useful expression when deriving aggregate consumption in steady state, later on.

Deriving \mathbb{Y} : Let households time related productivity be given by:

$$1 + z_t^{b^l+i} = \frac{y_{t+1}^{b^l+i}}{y_t^{b^l+i}} \quad (2.A.8)$$

Then a general expression for endowment present value at time t and its corresponding steady state version are given by:

$$\mathbb{Y}_t^{b^l} = \sum_{i=0}^{T-1} \frac{y_{t+i}^{b^l+i}}{\prod_{j=0}^{i-1} (1 + r_{t+b+j})} = \sum_{i=0}^{T-1} y_t^{b^l+i} \prod_{j=0}^{i-1} \frac{1 + z_{t+j}}{1 + r_{t+j}} \quad (2.A.9)$$

$$\text{In steady state: } \mathbb{Y}_t^{b^l} = \sum_{i=0}^{T-1} y_t^{b^l+i} \left(\frac{1 + z}{1 + r} \right)^i \quad (2.A.10)$$

For now we solve the model with a generic endowment path. Later on we will use a more specific expression for it, in order to find explicit expressions for the derivatives of the real interest rate w.r.t. age milestones.

We express household consumption at age $b^l + i$ in a similar way, by using expressions (2.A.3) and (2.A.6). It depends on the present value of income from age $b^l + i$ until the end of life, on expected real interest rate path, but now takes into account the real interest rate and net assets from previous year.

$$c_t^{b^l+i} = \frac{\mathbb{Y}_t^{b^l+i} - (1 + r_{t-1})b_{t-1}^{b^l+i-1}}{f(\beta_{r_t}, T - i)} \quad (2.A.11)$$

The borrowing level of an agent with age $b^l + i$ at time t , $b_t^{b^l+i}$ can be determined recursively using budget constraints (2.2) and (2.3), and using consumption expression (2.A.11). A simplified expression using determined past and present consumption and endowment, with recursion on the second budget constraint (2.3) is given by:

$$b_t^{b^l+i} = \mathbb{C}_{-t}^{b^l, b^l+i} - \mathbb{Y}_{-t}^{b^l, b^l+i} \quad (2.A.12)$$

where $\mathbb{C}_{-t}^{l,h}$ is the present value of past and present consumption of an agent with

age h at time t , from age l , and the same for endowment, given by expressions:

$$\mathbb{C}_{-t}^{l,h} = \mathbb{E}_t \sum_{j=0}^{h-l} \left(c_{t-j}^{h-j} \prod_{k=1}^j (1 + r_{t-k}) \right) \text{ and } \mathbb{Y}_{-t}^{l,h} = \mathbb{E}_t \sum_{j=0}^{h-l} \left(y_{t-j}^{h-j} \prod_{k=1}^j (1 + r_{t-k}) \right) \quad (2.A.13)$$

Equation (2.A.12) means that the borrowing level of an agent with age $b^l + i$ at time t is equal to the present values of past and present consumption and endowment levels since the beginning of his economic life, at age b^l . The term $(1 + r_{t-1})b_{t-1}^{b^l+i-1}$ is then determined by past endowment and consumption paths of an household.

Those tools will be useful to derive the transition dynamics of the model, namely when age milestones are changed.

To summarize, we derived expressions for consumption and borrowing levels of households with any age $b^l + i$ at time t . Consumption $c_t^{b^l+i}$ is expressed in terms of future income, expected future real interest rate path, and loans from previous year. And borrowing level at age $b^l + i$ and time t , $b_t^{b^l+i}$, is expressed in terms of present and past consumption, income and interest rate levels. c_t^i and b_t^i are then completely determined given an expected real interest rate path that solves loan market equilibrium, or excess borrowing equal to zero at any time t .

Assuming that the size of each generation at time t with age $b^l + i$ is N_t^i we define adulthood growth rate by $1 + g_t = \frac{N_t^{b^l}}{N_{t-1}^{b^l}} = \frac{N_t^{b^l+i}}{N_t^{b^l+1}}$. Equilibrium in the bond market is given by:

$$\forall t : B_t^{b^l} = 0 \quad (2.A.14)$$

where $B_t^{b^l}$ is excess borrowing normalized to the size of generation of age b^l :

$$B_t^{b^l} = \frac{1}{N_t^{b^l}} \sum_{i=0}^{T-1} N_t^{b^l+i} b_t^{b^l+i} = \sum_{i=0}^{T-1} \frac{N_t^{b^l+i}}{N_t^{b^l}} b_t^{b^l+i} = \sum_{i=0}^{T-1} \frac{b_t^{b^l+i}}{\prod_{k=0}^{i-1} (1 + g_{t-k})} \quad (2.A.15)$$

Excess borrowing $B_t^{b^l}$ can be expressed in terms of aggregate endowment and consumption at time t , and excess borrowing at $t - 1$, by replacing the second budget constraint (2.3) in (2.A.15):

$$B_t^{b^l} = \sum_{i=0}^{T-1} \frac{c_t^{b^l+i} - y_t^{b^l+i} + (1+r_{t-1})b_{t-1}^{b^l+i-1}}{\prod_{k=0}^{i-1}(1+g_{t-k})} = C_t^{b^l} - Y_t^{b^l} + \left(\frac{1+r_{t-1}}{1+g_t} \right) B_{t-1}^{b^l} \quad (2.A.16)$$

where aggregate endowment normalized to the size of generation b^l at time t is expressed by:

$$Y_t^{b^l} = \sum_{i=0}^{T-1} \frac{y_t^{b^l+i}}{\prod_{j=0}^{i-1}(1+g_{t-j})} \quad (2.A.17)$$

$$\text{steady state, } Y_t^{b^l} = \sum_{i=0}^{T-1} y_t^{b^l+i} \frac{1}{(1+g)^i} \quad (2.A.18)$$

Note that directly from endowment present value and aggregate expressions (2.A.10) and (2.A.18):

$$\frac{1+z}{1+r} = \frac{1}{1+g} \Leftrightarrow 1+r = (1+g)(1+z) = 1+r_{gz} \quad (2.A.19)$$

$$\Rightarrow \mathbb{Y}(r_{gz}) = Y \quad (2.A.20)$$

Moreover, aggregate consumption in period t normalized to the size of the youngest generation with age b^l is given by:

$$C_t^{b^l} = \sum_{i=0}^{T-1} \frac{c_t^{b^l+i}}{\prod_{j=0}^{i-1}(1+g_{t-j})} \quad (2.A.21)$$

$$\text{in steady state, } C_t^{b^l} = \sum_{i=0}^{T-1} \frac{c_t^{b^l+i}}{(1+g)^i} \quad (2.A.22)$$

Note that in steady state, $c_t^{b^l+i}$ can be expressed in terms of $c_t^{b^l}$:

$$c_t^{b^l+i} = c_{t-i}^{b^l} [\beta_r(1+r)]^i \quad (2.A.23)$$

and that $c_{t-i}^{b^l}$ can be expressed in terms of $c_t^{b^l}$:

$$c_{t-i}^{b^l} = \frac{\mathbb{Y}_{t-i}^{b^l}}{f(\beta_r, T)} = \frac{1}{(1+z)^i} \frac{\mathbb{Y}_t^{b^l}}{f(\beta_r, T)} = \frac{c_t^{b^l}}{(1+z)^i} \quad (2.A.24)$$

$$\Rightarrow c_t^{b^l+i} = c_{t-i}^{b^l} [\beta_r(1+r)]^i = \quad (2.A.25)$$

$$= c_t^{b^l} \left(\frac{\beta_r(1+r)}{1+z} \right)^i \quad (2.A.26)$$

Then, in steady state:

$$C_t^{b^l} = \sum_{i=0}^{T-1} \frac{c_t^{b^l+i}}{(1+g)^i} = c_t^{b^l} \sum_{i=0}^{T-1} \left[\frac{\beta_r(1+r)}{(1+g)(1+z)} \right]^i = c_t^{b^l} f(\gamma_r, T) = \mathbb{Y}_t^{b^l} \frac{f(\gamma_r, T)}{f(\beta_r, T)} \quad (2.A.27)$$

where $\gamma_r = \frac{\beta_r(1+r)}{(1+g)(1+z)}$. Note that if $r = r_{gz}$ the $\gamma_r = \beta_r$. We have seen that $\mathbb{Y}(r_{gz}) = Y$, then in steady state:

$$C(r_{gz}) = Y \quad (2.A.28)$$

This is an important point in proof of the continuity of excess borrowing in proposition 2.

We define an equilibrium as a set of processes $\{c_t^i, b_t^i, r_t\} \forall i \in [b^l, d^L]$ that solve (2.1) subject to (2.2),(2.3),(2.4),(3.A.13), and (2.A.14), given an exogenous process for $\{b_t^l, m_t^l, o_t^l, d_t^L, g_t, z_t, \rho_t^i\}, \forall t$. Inspired by Auerbach and Kotlikoff [3], the effect of changing age milestones at time t on existing cohorts is the same as if they were born again, behaving like members of a new generation but with a shorter life expectancy, and initial assets resulting from prior accumulation.

2.B Proposition 2: *Excess Borrowing* Steady State Properties

(i) The steady state expression for excess borrowing normalized to the size of the younger generation in the model, with age b^l , is derived directly from (2.A.16):

$$B_t^{b^l}(r, x) = \begin{cases} \frac{1+r_{gz}}{r-r_{gz}}(Y_t^{b^l} - C_t^{b^l}) & \text{for } r \neq r_{gz} \\ -(1+r_{gz})\frac{\partial C^{b^l}}{\partial r} & \text{for } r = r_{gz} \end{cases}, \text{ where } 1+r_{gz} = (1+g)(1+z) \quad (2.B.1)$$

The lower line of the expression ensures that $B^{b^l}(r, x)$ is continuously differentiable for $r \in]-1, +\infty[$, in particular for $r = r_{gz}$.

Proof (i): in steady state $r_t = r$ and so on for g_t and z_t :

$$B_t = C_t - Y_t - (1 + r_{t-1})B_{t-1} = C_t - Y_t - (1 + r_{t-1})\frac{B_t}{(1+g)(1+z)} \quad (2.B.2)$$

$$\Leftrightarrow B \left[1 - \frac{1+r}{(1+g)(1+z)} \right] = C - Y \quad (2.B.3)$$

$$\Leftrightarrow B = \frac{1+r_{gz}}{r-r_{gz}}(Y - C) \blacksquare \quad (2.B.4)$$

Continuity and differentiability of excess borrowing when $r = r_{gz}$

Aggregate endowment is a constant function of r , and aggregate consumption is continuous and differentiable for $r \in]-1, +\infty[$. Then we just have to prove the continuity and differentiability of steady state excess borrowing for $r = r_{gz}$. For that we use equation (2.A.28), or $Y = C(r_{gz})$:

$$\lim_{r \rightarrow r_{gz}} B(r) = \lim_{r \rightarrow r_{gz}} \frac{1+r_{gz}}{r-r_{gz}} [Y - C(r)] \quad (2.B.5)$$

$$= -(1+r_{gz}) \lim_{r \rightarrow r_{gz}} \frac{C(r) - C(r_{gz})}{r-r_{gz}} \quad (2.B.6)$$

$$= -(1+r_{gz}) \frac{\partial C}{\partial r}(r_{gz}) \quad (2.B.7)$$

Then $B(r_{gz}) = -(1+r_{gz})\frac{\partial C}{\partial r}(r_{gz})$ ensures the continuity of excess borrowing for

$r \in]-1, +\infty[$ in particular for $r = r_{gz}$. Likewise, $B'(r_{gz}) = -(1+r_{gz})C''(r_{gz})$ ensures the differentiability of excess borrowing for $r \in]1, +\infty[$ in particular for $r = r_{gz}$ ■.

(ii) *Existence and properties of a solution $B=0$: $B(r, x) = 0$ has at least one solution if the no endowment retirement period duration $\equiv o$ is lower than the elasticity of inter-temporal substitution times the model duration, or $\frac{1}{\sigma} > \frac{o}{T-1}$, and the duration of the initial no-endowment borrowing period $\equiv b$ is strictly lower than $T-1$. Moreover, $\frac{\partial B}{\partial r}(r) < 0$, at least for one solution r solving $B(r) = 0$.*

Proof: We prove that if this condition is verified then excess borrowing tends to $+\infty$ when r tends to -1^+ , and to $-\infty$ when r tends to $+\infty$. Then, because of the continuity and differentiability of excess borrowing there exists at least one solution r such that $B(r) = 0$, where $B'(r) \geq 0$:

With endowment levels equal to zero at the beginning and end of life, aggregate and present value expressions can be given by:

$$Y_t^{bl} = \frac{1}{(1+g)^i} \sum_{i=0}^{m-1} \frac{y_t^{m^l+i}}{(1+g)^i} \quad (2.B.8)$$

$$\mathbb{Y}_t^{bl} = \left(\frac{1+z}{1+r} \right)^b \sum_{i=0}^{m-1} y_t^{m^l+i} \left(\frac{1+z}{1+r} \right)^i \quad (2.B.9)$$

where b is the duration of the initial no-endowment period, m the duration of the period while $y^i > 0$, and m^l the age where the endowment period begins.

(i) $B(r = -1^+) > 0$? Because the term $\frac{1+r_{gz}}{r-r_{gz}} < 0$ if $r = -1^+$, and Y is constant in terms of r , then it is sufficient that $C(-1^+) = \mathbb{Y}(-1^+) \frac{f(\gamma_{-1^+}, T)}{f(\beta_{-1^+}, T)} = +\infty$. If $\frac{1}{\sigma} > 2$ then $\beta_{r=-1^+} = \beta_{\frac{1}{\sigma}}(1 + (-1^+))^{\frac{1}{\sigma}-1} = 0^+ \Rightarrow f(\beta_{-1^+}, T) = 1 \Rightarrow C(-1^+) = +\infty \rightarrow B(-1^+) = +\infty$. If $\frac{1}{\sigma} < 2$ then $f(\beta_{-1^+}, T) = +\infty$. $f(\beta_r, T) = \beta_r^{T-1} f(\beta_r^{-1}, T)$. $f(\beta_{-1^+}^{-1}, T) = 1 \Rightarrow f(\beta_{-1^+}, T) \rightarrow \beta^{\frac{T-1}{\sigma}}(1+r)^{(T-1)(\frac{1}{\sigma}-1)} \Rightarrow C(-1^+) \rightarrow \frac{y_t^l (1+r)^{-b} (1+r)^{-(m-1)}}{\beta^{\frac{T-1}{\sigma}} (1+r)^{(T-1)(\frac{1}{\sigma}-1)}} = y_t^l \beta^{\frac{1-T}{\sigma}} (1+r)^{T-b-m-\frac{T-1}{\sigma}} = +\infty$ if $T-b-m-\frac{T-1}{\sigma} = o - \frac{T-1}{\sigma} < 0 \Leftrightarrow \frac{o}{T-1} < \frac{1}{\sigma}$.

(ii) $B(r = +\infty) < 0$? Because the term $\frac{1+r_{gz}}{r-r_{gz}} > 0$ if $r = +\infty$, and Y is constant

in terms of r , then it is sufficient that $C(+\infty) = \mathbb{Y}(+\infty) \frac{f(\gamma_{+\infty}, T)}{f(\beta_{+\infty}, T)} = +\infty$. If $\frac{1}{\sigma} < 2$ then $f(\beta_{+\infty}) = 0 \Rightarrow f(\beta_{+\infty}, T) = 1 \Rightarrow C(+\infty) \rightarrow \frac{(1+r)^{T-1}}{(1+r)^b} = +\infty$ if $T - 1 > b$, a plausible assumption. If $\frac{1}{\sigma} > 2$ then $f(\beta_{+\infty}, T) = +\infty$. $f(\beta_r, T) = \beta_r^{T-1} f(\beta_r^{-1}, T)$. $f(\beta_{+\infty}^{-1}, T) = 1 \Rightarrow f(\beta_{+\infty}, T) \rightarrow \beta_{+\infty}^{T-1} \Rightarrow C(+\infty) \rightarrow y_t^b (1+r)^{-b} \frac{\beta_{+\infty}^{T-1} (1+r)^{T-1}}{\beta_{+\infty}^{T-1}} = +\infty$ if $T - 1 > b$ ■

If the ratio of the retirement period to the total duration of the model is lower than the elasticity of inter-temporal substitution $\frac{1}{\sigma}$, and $T > b - 1$, then there exists a steady state equilibrium solution for r , where $B(r) = 0$, and excess borrowing is a decreasing function of r around that solution.

(iii) *The derivative of the real interest rate r with respect to any parameter of the model x can be expressed by the following expression when loan market is in equilibrium:*

$$\frac{\partial r}{\partial x} \equiv r_x = -\frac{B_x}{B_r} = -\frac{Y_x - C_x}{Y_r - C_r} = -\frac{\log_x Y - \log_x C}{\log_r Y - \log_r C} \quad (2.B.10)$$

Proof: We use the partial derivatives of steady state excess borrowing with respect to r and to any parameter x , from the steady state version of expression (2.A.16), to formulate the derivative of the natural rate of interest with respect to parameter x (in particular an age milestone v^i), assuming equilibrium in the loan market in steady state, or $B^{bl}(r, x) = 0$:

$$B(r, x) = 0 \Rightarrow r_x = \frac{dr}{dx}(r, x) = -\frac{\frac{\partial B}{\partial x}(r, x)}{\frac{\partial B}{\partial r}(r, x)} \Leftrightarrow r_x = -\frac{B_x}{B_r}(r, x) \quad (2.B.11)$$

To inspect how the equilibrium real interest rate changes with respect to change of any exogenous parameter of the model x , we derive closed form expressions for $\frac{dr}{dx_i}(r, x_i, x_{-i})$, where r solves $B^{bl}(r, x) = 0$ given $x \equiv (x_i, x_{-i}) \in \mathbb{R}^n$, where n is the number of exogenous parameters of the model, and $x_i \in V \equiv$ all exogenous parameters of the model, in particular the age milestones with the exception of s^l .

We use the fact that excess borrowing must remain equal to zero after a change in x_i . In other words, we need to derive by how much the equilibrium real interest rate must change in order to compensate for the impact of a change in x^i on excess borrowing so that it remains constant, and equal zero given loan market equilibrium:

$$dB(r, x) = \frac{\partial B}{\partial x^i} dx^i + \frac{\partial B}{\partial r} dr = 0 \Leftrightarrow \quad (2.B.12)$$

$$r_{x^i}(r, x) = \frac{dr}{dx^i} = -\frac{\frac{\partial B}{\partial x^i}}{\frac{\partial B}{\partial r}} = -\frac{B_{x^i}}{B_r}(r, x), \text{ for } r \text{ such that } B(r, x) = 0 \quad (2.B.13)$$

Note that

$$B_x \equiv \frac{\partial B}{\partial x} = \frac{\partial}{\partial x} \left(\frac{1 + r_{gz}}{r - r_{gz}} \right) (Y - C) + \frac{1 + r_{gz}}{r - r_{gz}} \left(\frac{\partial Y}{\partial x} - \frac{\partial C}{\partial x} \right) \quad (2.B.14)$$

In equilibrium $B = 0 \Rightarrow Y = C$, so the previous expression can be simplified to:

$$B_x = \frac{1 + r_{gz}}{r - r_{gz}} \left(\frac{\partial Y}{\partial x} - \frac{\partial C}{\partial x} \right) = \frac{1 + r_{gz}}{r - r_{gz}} Y \left(\frac{Y_x}{Y} - \frac{C}{Y} \frac{C_x}{C} \right) = \frac{1 + r_{gz}}{r - r_{gz}} Y (\log_x Y - \log_x C) \quad (2.B.15)$$

$$\text{where } \log_x Y = \frac{\partial \log Y}{\partial x} = \frac{1}{Y} \frac{\partial Y}{\partial x} = \frac{Y_x}{Y} \Leftrightarrow Y_x = Y \log_x Y \quad (2.B.16)$$

And from where we can directly derive a simplified expression for the partial derivative of excess borrowing with respect to the real interest rate, in equilibrium, given by²²:

$$B_r = \frac{1 + r_{gz}}{r - r_{gz}} Y (\log_r Y - \log_r C) = -\frac{1 + r_{gz}}{r - r_{gz}} Y \log_r C \quad (2.B.17)$$

By combining expressions (2.B.15) and (2.B.17) an expression for r_x is given by:

$$r_x \equiv \frac{\partial r}{\partial x} = -\frac{B_x}{B_r} = \frac{\log_x Y - \log_x C}{\log_r C} \quad (2.B.18)$$

²²Because in this model aggregate endowment is independent of the real interest rate, $\log_r Y = 0$

2.C Proposition 3: Real Interest Rate Derivatives w.r.t age parameters

For a sufficiently generic households' endowment path expressed by:

$$y_t^i > 0 \text{ for } i \in [m^l, m^h] \quad (2.C.1)$$

$$y_t^i = 0 \text{ for } i \in [b^l, m^l[\cup]m^h, d^L] \quad (2.C.2)$$

where time and age related productivity growth rates, for $i \in [m^l, m^h]$, respectively given by $1 + z_t^i = \frac{y_{t+1}^i}{y_t^i}$, and $1 + \rho_t^i = \frac{y_{t+1}^{i+1}}{y_t^i}$, steady state expressions for aggregate endowment and consumption are respectively given by:

$$Y_t^{b^l} = \frac{y_t^{m^l}}{(1+g)^b} f\left(\frac{1+\rho}{1+g}, m\right) \quad (2.C.3)$$

$$C_t^{b^l} = \mathbb{Y}_t^{b^l} \frac{f(\gamma, T)}{f(\beta_r, T)} \quad (2.C.4)$$

and $\mathbb{Y}_t^{b^l}$ given by:

$$\mathbb{Y}_t^{b^l} = \frac{y_t^{m^l}}{(1+r_z)^b} f\left(\frac{1+\rho}{1+r_z}, m\right) \quad (2.C.5)$$

where periods duration are defined by $b = m^l - b^l$, $m = o^l - m^l$, where $o^l = m^l + 1$. In addition, the partial derivative of excess borrowing with respect to the real interest rate $B_r(r, v) \equiv B_r$ is the denominator common to the derivatives of the real interest rate with respect to all exogenous parameters of the model, where $\text{Log}_r C$ can be expressed by:

$$\text{Log}_r C = \text{Log}_r \mathbb{Y} + \text{Log}_r f(\gamma, T) - \text{Log}_r f(\beta_r, T) \quad (2.C.6)$$

where the algebraic expressions of each one of the terms are given by:

$$Log_r \mathbb{Y} = \frac{1}{1+r} \left(-b + \frac{m}{(1+r_{z\rho})^m - 1} - \frac{1}{r_{z\rho}} \right) \quad (2.C.7)$$

$$Log_r f(\gamma_r, T) = \frac{1}{1+r} \left(\frac{1}{\sigma} \right) \left(T \frac{\gamma_r^T}{\gamma_r^T - 1} - \frac{\gamma_r}{\gamma_r - 1} \right) \quad (2.C.8)$$

$$Log_r f(\beta_r, T) = \frac{1}{1+r} \left(\frac{1-\sigma}{\sigma} \right) \left(T \frac{\beta_r^T}{\beta_r^T - 1} - \frac{\beta_r}{\beta_r - 1} \right) = 0 \text{ for } \sigma = 1 \quad (2.C.9)$$

where $1 + r_{z\rho} = \frac{1+r}{(1+z)(1+\rho)}$. Now that we have closed-form expression for aggregate consumption and endowment, we start by expressing the derivatives of excess borrowing with respect to the durations d of the main periods of the model, $b = m^l - b^l$, $m = o^l - m^l$ and $T = L - b^l + 1$, (the retirement period is here a dependent variable), for algebraic simplification, and because we can express the partial derivatives of B and r w.r.t. age milestones as a linear combination of durations due to the above linear relation between those two sets of parameters:

$$B_{b^l} = B_b b_{b^l} + B_m m_{b^l} + B_T T_{b^l} = -B_b + 0 - B_T = -B_b - B_T \quad (2.C.10)$$

$$B_{m^l} = B^b - B^m \quad (2.C.11)$$

$$B_{o^l} = B^m \quad (2.C.12)$$

$$B_L = B_T \quad (2.C.13)$$

Same relations valid for r_{v_i} .

Deriving $\log_d Y$, where $d \in \{b, m, T\}$:

$$\log_d Y = \log_d \left[\frac{y^{m^l}}{(1+g)^b} f \left(\frac{1+\rho}{1+g}, m \right) \right] \quad (2.C.14)$$

$$= -b_d \log(1+g) + \log_d f \left(\frac{1+\rho}{1+g}, m \right) \quad (2.C.15)$$

$$\text{then } \log_b Y = -\log(1+g) \quad (2.C.16)$$

$$\log_m Y = \log_m f \left(\frac{1+\rho}{1+g}, m \right) = H \left(\frac{1+\rho}{1+g}, m \right) \quad (2.C.17)$$

$$\text{and } \log_T Y = 0 \quad (2.C.18)$$

where $H(x, m) = \frac{h(x^{-m})}{m}$, and $h(y) = \frac{\log y}{y-1} = \frac{\log y - \log 1}{y-1} > 0$, naturally²³. Note that $H_x \equiv \frac{\partial H}{\partial x} > 0$. Proof $H_x > 0$:

(i) First we prove that $h'(y) < 0$: $h'(y) = \frac{1 - \frac{1}{y} - \log y}{(y-1)^2} < 0 \Leftrightarrow 1 - \frac{1}{y} - \log y = g(y) < 0$. $g'(y) = \frac{1-y}{y^2} \Rightarrow g'(y) > 0$ if $y < 1$, $g'(y) < 0$ if $y > 1$, and $g'(1) = 0$. So $g(y)$, and $h'(y)$ are always strictly negative for $y \neq 1$, with a maximum equal to 0 for $y = 1$.

(ii) $H_x(x, m) = \frac{1}{m} h'(x^{-m})(x^{-m})' = -x^{-m-1} h'(x^{-m}) > 0 \forall m \in \mathbb{Z}$ ■

Deriving $\log_d C$:

$$\log_d C = \log_d \left[\mathbb{Y} \frac{f(\gamma_r, T)}{f(\beta_r, T)} \right] \quad (2.C.19)$$

$$= -b_d \log(1+r_z) + \log_d f \left(\frac{1+\rho}{1+r_z}, m \right) + \log_d f(\gamma_r, T) - \log_d f(\beta_r, T) \quad (2.C.20)$$

$$\text{then } \log_b C = -\log(1+r_z) \quad (2.C.21)$$

$$\log_m C = H \left(\frac{1+\rho}{1+r_z}, m \right) \quad (2.C.22)$$

$$\text{and } \log_T C = H(\gamma_r, T) - H(\beta_r, T) \quad (2.C.23)$$

Let $\triangle f(x, y) = \frac{f(x)-f(y)}{x-y}$, and note that $f' > 0 \Rightarrow \triangle f > 0$, directly from the *Mean*

²³ $h(y) > 0$ by applying the *Mean Value Theorem*

CHAPTER 2. AGE MILESTONES AND LOW INTEREST RATES

Value Theorem. For notation purposes let $H^m(x) \equiv H(x, m)$, $H_x^m \equiv H_x(x, m)$, and $\Delta H^m(x, y) = \frac{H^m(x) - H^m(y)}{x - y} > 0 \forall m \in \mathbb{Z}$, since $H_x^m > 0$. The derivatives of excess borrowing w.r.t. periods duration d are given by:

$$B_b = Y \frac{1 + r_{gz}}{r - r_{gz}} (\log_b Y - \log_b C) = Y(1 + r_{gz}) \Delta \log(1 + r, 1 + r_{gz}) \quad (2.C.24)$$

$$= Y \beta_r \Delta \log(\gamma_r, \beta_r) > 0 \quad (2.C.25)$$

$$B_m = Y \frac{1 + \rho}{1 + r_z} \Delta H^m \left(\frac{1 + \rho}{1 + g}, \frac{1 + \rho}{1 + r_z} \right) > 0 \quad (2.C.26)$$

$$B_T = -Y \beta_r \Delta H^T(\gamma_r, \beta_r) < 0 \quad (2.C.27)$$

$$(2.C.28)$$

Derivatives of Excess Borrowing w.r.t. age milestones:

$$B_{b^l} = -B_b - B_t = Y \beta_r [\Delta H^T(\gamma_r, \beta_r) - \Delta \log(\gamma_r, \beta_r)] \quad (2.C.29)$$

$$= -Y \beta_r \Delta H^{-T}(\gamma_r, \beta_r) < 0 \quad (2.C.30)$$

$$B_{m^l} = B_b - B_m = Y \frac{1 + \rho}{1 + r_z} \Delta H^{-m} \left(\frac{1 + \rho}{1 + g}, \frac{1 + \rho}{1 + r_z} \right) > 0 \quad (2.C.31)$$

$$B_{o^l} = B_m > 0 \quad (2.C.32)$$

$$B_L = B_T < 0 \quad (2.C.33)$$

Derivatives r with respect to durations:

$$r_b = \frac{\log_b Y - \log_b C}{\log_r C} = \frac{1}{\log_r C} \log \left(\frac{1 + r}{1 + r_{gz}} \right) = \frac{1}{\log_r C} (\log(\gamma_r) - \log(\beta_r)) > 0 \quad (2.C.34)$$

$$r_m = \frac{\log_m Y - \log_m C}{\log_r C} = \frac{1}{\log_r C} \left[H^T \left(\frac{1 + \rho}{1 + g} \right) - H^T \left(\frac{1 + \rho}{1 + r_z} \right) \right] > 0 \quad (2.C.35)$$

$$r_T = \frac{\log_T Y - \log_T C}{\log_r C} = -\frac{1}{\log_r C} [H^T(\gamma_r) - H^T(\beta_r)] < 0 \quad (2.C.36)$$

Derivatives of r with respect to age milestones, and their sign, assuming that $B_r < 0$:

$$r_{b^l} = -r_b - r_T = -\frac{1}{\log_r C} [H^{-T}(\gamma_r) - H^{-T}(\beta_r)] < 0 \quad (2.C.37)$$

$$r_{m^l} = r_b - r_m = \frac{1}{\log_r C} \left[H^{-m} \left(\frac{1+\rho}{1+g} \right) - H^{-m} \left(\frac{1+\rho}{1+r_z} \right) \right] > 0 \quad (2.C.38)$$

$$r_{o^l} = r_m = \frac{1}{\log_r C} \left[H^m \left(\frac{1+\rho}{1+g} \right) - H^m \left(\frac{1+\rho}{1+r_z} \right) \right] > 0 \quad (2.C.39)$$

$$r_L = r_T = -\frac{1}{\log_r C} [H^T(\gamma_r) - H^T(\beta_r)] < 0 \quad (2.C.40)$$

2.D OLG model with endogenous output and capital:

Household problem

The same household objective function, and the budget constraints where wages and capital are introduced have expressions given by:

$$\max_{c_{t+i}^{b^l+i}} E_t \left\{ \sum_{i=0}^{T-1} \beta^i U(c_{t+i}^{b^l+i}) \right\} \quad (2.D.1)$$

$$\text{s.t. } c_t^{b^l} = w_t^{b^l} l_t^{b^l} - a_t^{b^l} \quad (2.D.2)$$

$$c_{t+i}^{b^l+i} = w_{t+i}^{b^l+i} l_{t+i}^{b^l+i} + (1 + r_{t+i-1}) a_{t+i-1}^{b^l+i-1} - a_{t+i}^{b^l+i} \quad (2.D.3)$$

$$c_{t+T-1}^{o^h} = w_{t+T-1}^{o^h} l_{t+T-1}^{o^h} + (1 + r_{t+T-2}) a_{t+T-2}^{o^h-1} \quad (2.D.4)$$

where utility of consumption $U(c) = \frac{c^{\sigma-1}}{1-\sigma}$, and elasticity of inter-temporal substitution $EIS = \frac{1}{\sigma}$. Assets $a_t = k_t - b_t$ are composed by capital k_t that households rent to firms, and loans to other households $-b_t$. The capital portion of assets is always positive but the loans to other households $-b_t$ can be positive or negative. In any case, agents are called borrowers when $a_t < 0$ and savers otherwise. While employed each household is given an exogenous annual labor endowment that is as-

sumed to increase with work experience at a stable rate²⁴. The expression for labor endowment at age i is given by:

$$l_t^{m^l+i} = l_t^{m^l} (1 + \rho)^i \quad (2.D.5)$$

Without loss of generality we assume that the labor endowment in the beginning of the working period $l^{m^l} = 1$.

Regarding households' asset composition $a_t = k_t - b_t$, we can rewrite the budget constraints in terms of loans and capital:

$$c_{t+i}^{b^l+i} = w_{t+i}^{b^l+i} l_{t+i}^{b^l+i} + \left[(1 - \delta) k_{t+i-1}^{b^l+i-1} + r_{t+i-1}^k k_{t+i-1}^{b^l+i-1} - (1 + r_{t+i-1}) b_{t+i-1}^{b^l+i-1} \right] - \left[k_{t+i}^{b^l+i} - b_{t+i}^{b^l+i} \right] \quad (2.D.6)$$

From the first order conditions of k_t and b_t the *No-Arbitrage Condition (NAC)* is given by:

$$r_t^k = r_t + \delta \quad (2.D.7)$$

Equilibrium expressions for consumption and asset supply are derived as in section 2. From households' budget constraints with capital we derive a present value household budget constraint, now given by:

$$\mathbb{C}_t^{b^l+i} = \mathbb{W}_t^{b^l+i} + (1 + r_{t-1}) a_{t-1}^{b^l+i-1} \quad (2.D.8)$$

$$\text{where } \mathbb{W}_t^{b^l+i} = \mathbb{E}_t \sum_{j=0}^{m^h-i} \frac{w_{t+j} l_{t+j}^{b^l+j}}{\prod_{k=0}^{j-1} (1 + r_{t+k})} \quad (2.D.9)$$

which means that the present value of consumption from age i until life expectancy L^e is equal to the present value of labor income during the same period plus the assets and respective interests from the previous age $i - 1$. In the model with capital

²⁴We use this assumption to ensure algebra tractability.

endowment y_t^i is replaced by labor income $w_t l_t^i$. From the same Euler equation (2.D.10) we find also the same expression for consumption of an household at time t and age $b^l + i$ as a function of its consumption present value until L^e :

$$\mathbb{E}_t \frac{c_{t+1}^{b^l+i+1}}{1+r_t} = \beta_{r_t} c_t^{b^l+i} \Rightarrow \quad (2.D.10)$$

$$\Rightarrow \mathbb{C}_t^{b^l+i} = c_t^{b^l+i} f(\beta_{r_t}, T-i) \quad (2.D.11)$$

where $f(\beta_{r_t}, T-i) = \mathbb{E}_t \sum_{j=0}^{T-i-1} \prod_{k=1}^j \beta_{r_{t+k-1}}$. The equilibrium expressions for household consumption and assets have the same expressions as in section 2 where lending to other households $-b_t^i$ that can be positive or negative is replaced by assets a_t^i . For $i \neq 0$:

$$c_t^{b^l+i} = \frac{1}{f(\beta_{r_t}, T-i)} [\mathbb{W}_t^i + (1+r_{t-1})a_{t-1}^{i-1}] \quad (2.D.12)$$

$$a_t^{b^l+i} = w_t l_t^{b^l+i} - c_t^{b^l+i} + (1+r_{t-1})a_{t-1}^{b^l+i-1} \quad (2.D.13)$$

In the beginning of model's duration period, at age b^l , or for $i = 0$:

$$c_t^{b^l} = \frac{\mathbb{W}_t^{b^l}}{f(\beta_r, T)} \quad (2.D.14)$$

$$a_t^{b^l} = w_t l_t^{b^l} - c_t^{b^l} \quad (2.D.15)$$

Firms problem:

Firms are perfectly competitive and take prices are given. They hire labor and rent capital to maximize profits. Their profit maximization problem is given by:

$$\max_{y_t, k_t} y_t - w_t l_t - r_t^k k_t \quad (2.D.16)$$

$$\text{s.t. } y_t = Z_t l_t^\alpha k_t^{1-\alpha}, \text{ where } Z_t = (1+z)Z_{t-1} \quad (2.D.17)$$

Demand for labor and capital in equilibrium is respectively given by:

$$l_t = \alpha \frac{y_t}{w_t} \quad (2.D.18)$$

$$k_t = (1 - \alpha) \frac{y_t}{r_t^k} \quad (2.D.19)$$

By replacing the previous demand expressions in the production function we derive equilibrium expressions for output and wages at time t :

$$y_t = Z_t l_t^\alpha k_t^{1-\alpha} = l_t Z_t^{\frac{1}{\alpha}} \left(\frac{1-\alpha}{r_t^k} \right)^{\frac{1-\alpha}{\alpha}} \quad (2.D.20)$$

$$w_t = \alpha \frac{y_t}{l_t} = \alpha Z_t^{\frac{1}{\alpha}} \left(\frac{1-\alpha}{r_t^k} \right)^{\frac{1-\alpha}{\alpha}} \quad (2.D.21)$$

Equilibrium in the asset market

Same loan market equilibrium condition:

$$\forall t, L_t^T = B_t = 0 \Leftrightarrow A_t = K_t \quad (2.D.22)$$

$$\text{where } A_t^T = \frac{1}{N_t^{b^l}} \sum_{i=0}^{T-1} N_t^{b^l+i} a_t^{b^l+i} = \sum_{i=0}^{T-1} \frac{N_t^{b^l+i}}{N_t^{b^l}} a_t^{b^l+i} = \sum_{i=0}^{T-1} \frac{a_t^{b^l+i}}{\prod_{j=0}^{i-1} (1 + g_{t-j})} \quad (2.D.23)$$

$$K_t = (1 - \alpha) \frac{Y_t}{r_t^k} = \frac{1 - \alpha}{\alpha} \frac{W_t}{r_t + \delta} \quad (2.D.24)$$

$$W_t = w_t L_t = \frac{w_t l_t^{m^l}}{\prod_{i=0}^{b-1} (1 + g_{t-i})} \sum_{i=0}^{m-1} \prod_{j=0}^{i-1} \frac{1 + \rho^{b^l+j}}{1 + g_{t-j}} \quad (2.D.25)$$

An equilibrium is defined in the same way as in section 2, as a set of processes $\{c_t^i, a_t^i, r_t\}$ that solve (2.D.2),(2.D.3),(2.D.4),(2.D.10), and (2.D.22) $\forall i$, given an exogenous process for $\{g_t, b_t^l, m_t^l, o_t^l, L_t\}$.

Relevant expressions in steady state

Previous expressions for steady state equilibrium are algebraically more tractable.

Aggregate borrowing is continuous and differentiable for $r \in]-\delta, +\infty[$:

$$B_t^{bl} = \begin{cases} \frac{1+r_{gz}}{r-r_{gz}}(W_t^{bl} - C_t^{bl}) + K_t^{bl} & \text{for } r \neq r_{gz} \\ -(1+r_{gz})w_t(r_{gz})\frac{\partial C_t^{bl}}{\partial r}(r_{gz}) + K_t^{bl} & \text{for } r = r_{gz} \end{cases} \quad (2.D.26)$$

Where $1+r_{gz} = (1+g)(1+z_\alpha)$, and $1+z_\alpha = (1+z)^\frac{1}{\alpha}$. Aggregate steady state expressions of consumption, labor income and capital, normalized to the population size of the model's youngest generation and the total factor productivity at time t are given by:

$$C_t^{bl} = w_t l_t^{m^l} \left(\frac{1}{1+r_z} \right)^b f \left(\frac{1+\rho}{1+r_z}, m \right) \frac{f(\gamma, T)}{f(\beta_r, T)} \quad (2.D.27)$$

$$W_t^{bl} = w_t l_t^{m^l} \left(\frac{1}{1+g} \right)^b f \left(\frac{1+\rho}{1+g}, m \right) \quad (2.D.28)$$

$$K_t^{bl} = \frac{1-\alpha}{\alpha} \frac{W_t^{bl}}{r+\delta} = w_t l_t^{m^l} \left(\frac{1}{1+g} \right)^b f \left(\frac{1+\rho}{1+g}, m \right) \frac{1-\alpha}{\alpha} \frac{1}{r+\delta} \quad (2.D.29)$$

Deriving aggregate consumption with capital

Aggregate consumption with capital is derived in the same way as in section 3, only replacing the present value of the endowment path \mathbb{Y}_t^{bl} by the present value of labor income path \mathbb{W}_t^{bl} :

$$C_t^{bl} = \sum_{i=0}^{T-1} \frac{c_t^{bl+i}}{\prod_{j=0}^{i-1} (1+g_{t-j})} \quad (2.D.30)$$

Note that in steady state the consumption of an agent of age i at time t is given by the following expression:

$$c_t^{bl+i} = c_{t-i}^{bl} [\beta_r(1+r)]^i = c_t^{bl} \left[\frac{\beta_r(1+r)}{1+z_\alpha} \right]^i = c_t^{bl} \gamma^i \quad (2.D.31)$$

Note that $c_{t-i}^{bl} = \frac{\mathbb{W}_{t-i}^{bl}}{f(\beta_r, T)} = \frac{1}{(1+z_\alpha)^i} \frac{\mathbb{W}_t^{bl}}{f(\beta_r, T)} = \frac{c_t^{bl}}{(1+z_\alpha)^i}$. Then:

$$C_t^{bl} = \sum_{i=0}^{T-1} \frac{c_t^{bl+i}}{(1+g)^i} = c_t^{bl} \sum_{i=0}^{T-1} \left[\frac{\beta_r(1+r)}{(1+g)(1+z_\alpha)} \right]^i = c_t^{bl} f(\gamma, T) = \mathbb{W}_t^{bl} \frac{f(\gamma, T)}{f(\beta_r, T)} \quad (2.D.32)$$

where $\gamma = \frac{\beta_r(1+r)}{(1+g)(1+z_\alpha)}$. And where \mathbb{W}_t^{bl} is the steady state expression for the present value of an household labor income path at time t :

$$\mathbb{W}_t^{bl} = \left(\frac{1+z_\alpha}{1+r} \right)^b \sum_{i=0}^{m-1} \frac{w_t^{m^l+i} l_t^{m^l+i}}{(1+r)^i} = w_t^{m^l} l_t^{m^l} \left(\frac{1}{1+r_z} \right)^b f\left(\frac{1+\rho}{1+r_z}, m \right) \quad (2.D.33)$$

where $1+r_z = \frac{1+r}{1+z_\alpha}$.

Continuity of excess borrowing when $r = r_{gz}$

If $r = r_{gz}$ then $\beta_r = \gamma$, and $1+r_z = 1+g$. Consequently, and directly from expressions (2.D.27) and (2.D.28), if $r = r_{gz}$ then $C^{bl}(r) = W^{bl}(r)$. $\frac{C^{bl}}{w_t}(r_{gz}) = \frac{W^{bl}}{w_t}$, where $\frac{W^{bl}}{w_t} = l_t^{m^l} \left(\frac{1}{1+g} \right)^b f\left(\frac{1+\rho}{1+g}, m \right)$ is independent of the real interest rate r . Aggregate borrowing can be alternatively expressed, for $r \neq r_{gz}$ as:

$$B_t^{bl} = \frac{1+r_{gz}}{r-r_{gz}} (W_t^{bl} - C_t^{bl}) + K_t^{bl} \quad (2.D.34)$$

$$= K_t^{bl} - (1+r_{gz})w_t(r_{gz}) \frac{\frac{C^{bl}}{w_t}(r) - \frac{C^{bl}}{w_t}(r_{gz})}{r-r_{gz}} \quad (2.D.35)$$

Since $\frac{C^{bl}}{w_t}(r) = l_t^{m^l} \left(\frac{1}{1+r_z} \right)^b f\left(\frac{1+\rho}{1+r_z}, m \right) \frac{f(\gamma, T)}{f(\beta_r, T)}$ is differentiable in $r \in]-\delta, +\infty[$ the following limit exists:

$$\lim_{r \rightarrow r_{gz}} B_t^{bl}(r) = K^{bl}(r_{gz}) - (1+r_{gz})w_t(r_{gz}) \lim_{r \rightarrow r_{gz}} \frac{\frac{C^{bl}}{w_t}(r) - \frac{C^{bl}}{w_t}(r_{gz})}{r-r_{gz}} \quad (2.D.36)$$

$$= K^{bl}(r_{gz}) - (1+r_{gz})w_t(r_{gz}) \frac{\partial \frac{C^{bl}}{w_t}}{\partial r}(r_{gz}) \quad (2.D.37)$$

The steady state natural rate of interest r_n solves $B(r_n) = 0$. The properties of $B(r)$ and the derivatives of the natural rate of interest with respect to age milestones are derived using the same methodology as in section 3.

The third point of proposition 3 has the following adjusted version:

(ii) Existence and properties of a solution $B=0$: $B(r, x) = 0$ has at least one solution, if the duration of the initial no-endowment borrowing period $\equiv b$ is strictly lower than $T - 1$. Moreover, $\frac{\partial B}{\partial r}(r) < 0$, at least for one solution r solving $B(r) = 0$.

Proof: We prove that excess borrowing tends to $+\infty$ when r tends to $-\delta^+$, and to $-\infty$ when r tends to $+\infty$ if $b < T - 1$. Then, because of the continuity and differentiability of excess borrowing there exists at least one solution r such that $B(r) = 0$, where $B'(r) \geq 0$:

Chapter 3

Inequality and Real Interest Rates

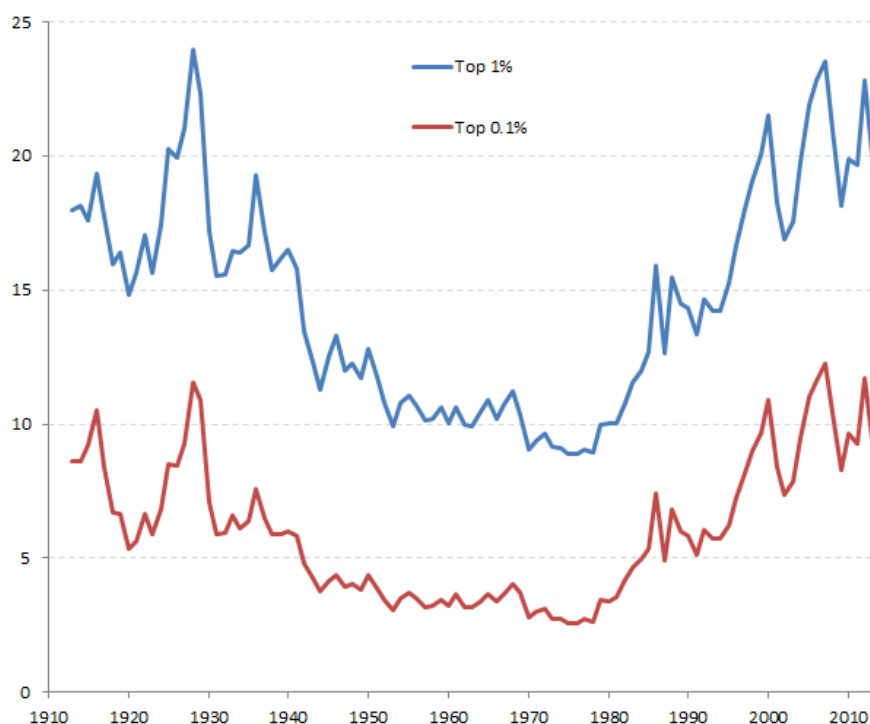
ABSTRACT

We explore the relation between *income inequality* and *real interest rates* based on the marginal borrowing and saving rates of an heterogeneous population with respect to lifetime income. We use an overlapping generations New Keynesian model with borrowing constraints and a bequest motive, to show how an increase of income inequality may trigger a permanent reduction of the real interest rate, (i) via a contraction of aggregate borrowing, when the marginal borrowing rate of the wealthier is lower than the one of the poorer with respect to income. (ii) We then show how an increase of inequality may trigger an expansion of savings through the channel of a bequest motive where generosity towards the next generation increases endogenously with lifetime income, so that the marginal savings rate of the wealthier is higher than the poorer.

3.1 Introduction

Income inequality has been increasing since the early 80's in relevant world economies (Piketty [33]). For example, in US the income share of the 1% wealthier increased from around 10% in 1980 to more than 20% in 2014 (Figure:3.1). During the same period the real interest rates have been decreasing to levels close to zero. In this

Figure 3.1: US income shares with capital gains 1913-2014: Top 0.1% and 1%



Source: Piketty [33]; The World Wealth and Income Database - WID

chapter we formally relate those two phenomena based on evidence from recent literature that the wealthier save marginally more and borrow marginally less than the poor, so that the net impact of an increase of income inequality would be an expansion of aggregate savings, together with a contraction of aggregate borrowing, thus dragging down the real interest rate.

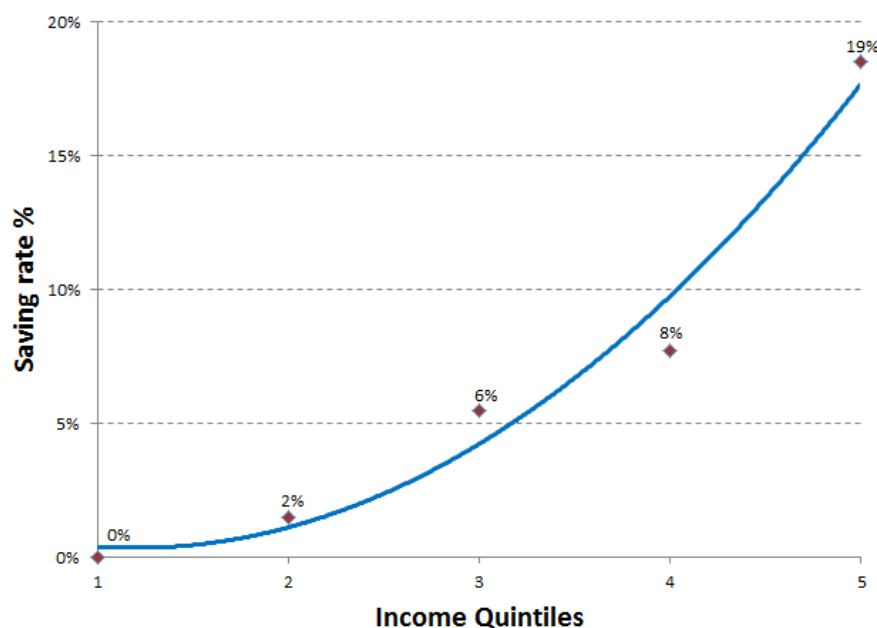
On the savings side, Dynan et al. [19] besides empirically validating the fact that *the rich save more* (Figure:3.2), also found evidence that the *marginal savings rate*

(MSR) is higher for high income households: In income quintiles 1 and 2 they estimate that MSR increases \$0.030 for each dollar of income increase, against \$0.429 for income quintiles 4 and 5. They also state that *higher savings rates for higher-income groups are consistent with an operative bequest motive*. Although, on another angle, bequests have also been considered in recent literature as a relevant source of income inequality (Galor [24], and Piketty [33]), in our work inequality is not endogenized, but it is the channel through which an endogenous bequest motive may become operative when parents expect their children to be less wealthy than themselves. So that when income inequality increases, high income type agents become even richer than the average, which further increases their endogenous generosity towards their children, whose expected wealth remains equal to population income mean, by increasing the level of bequests. The savings mechanism in our model relating inequality to real interest rates can then be summarized in the following way. Inequality turns on a bequest motive for the rich, and turns it off for the poor. The marginal savings rate with an operative bequest motive for the rich is greater than with an inoperative bequest motive for the poor, such that an increase of income inequality triggers an expansion of aggregate savings of the richer that prevails over the contraction of aggregate savings of the poor, resulting in a net expansion of total aggregate savings, that drags down the equilibrium real interest rate.

On the borrowing side, in a recent paper Mian and Sufi [32] found strong heterogeneity among *marginal propensities to borrow* (MPB) of households, by examining the impact of rising U.S. house prices on borrowing and spending from 2002 to 2006. They found evidence that the *rich* have a lower marginal propensity to borrow than the *poor*: Specifically, on average MPB increases by \$0.19 per dollar of home value increase, but by \$0.26 for the poorer households, and close to \$0 for the wealthier. Similarly, in a recent NBER working paper, Agarwal et al. [1] found empirical evidence that the *marginal propensity to borrow* is lower for higher credit scores, but in contrast with higher *marginal propensity to lend* (MPL) of the banks for higher

credit scores. Taking this into account, in a credit constrained environment, the *marginal borrowing rate* (MBR) should be driven by banks marginal propensity to lend to the poorer, and by the marginal propensity to borrow of the unconstrained richer. This is a relevant factor to look at when modeling a borrowing constraint later in the chapter.

Figure 3.2: Saving Rates and Income



Source: Dynan et al. [19], PSID

In this chapter we propose a formal framework to inspect the relation of increasing income inequality and persistently low real interest rates, based on decreasing marginal borrowing and increasing marginal saving rates with respect to income. So that an increase of income inequality triggers a net contraction of aggregate borrowing, as well as a net expansion of aggregate savings, resulting in a lower equilibrium real interest rate. We present a model where persistently low real interest rates are driven by increasing income inequality, by building and combining on some relevant topics in economic literature, namely on *Secular Stagnation* (Eggertsson and Mehrotra [21]), on the relation of marginal propensity to save and life-time income

(Dynan et al. [19]), on the relation of marginal propensity to borrow with income and wealth (Mian and Sufi [32]) and (Agarwal et al. [1]), on inequality (Galor [24], and Piketty [33]), and bequests (Barro and Sala-i Martin [5] and Blanchard and Fischer [7]).

In what follows, in the second section of this chapter we derive a formal general framework linking changes of real interest rates with increasing income inequality. We use the loan market equilibrium equation to derive an algebraic relation between (i) the partial derivative of the real interest rate with respect to the standard deviation of households' income distribution, (ii) and the marginal borrowing and saving rates of the wealthier and the poorer. We then use this relation to describe the mechanisms by which *MBR* decreases, and *MSR* increases with income, which respectively result in the contraction of aggregate borrowing, and the expansion of aggregate savings, when the standard deviation of income distribution increases. In the third section we formalize a more specific overlapping generations model with three periods based on the work of Eggertsson and Mehrotra [21], to explicitly describe the concepts presented before with a single agent type per generation, and observe that the real interest rate may decrease if the next generation is expected to be poorer.

In section 4, we introduce in the model income inequality among same generation households. We inspect how borrowing constraints alone can trigger a reduction of real interest rates when income inequality increases. Then, we use a *warm glow* bequest motive to start by showing that the equilibrium real interest rate can be persistently low, and materially lower than if the bequest motive is not operative. By adding income inequality into the model, and by assuming that generosity with respect to children is an exogenous positive function of income, we get an increasing marginal savings rate with income, causing the equilibrium real interest rate to decrease with an increase of income inequality. The mechanism is straightforward.

If the rich get richer, then bequests increase more than proportionally to income: Bequests primarily increase because income increases, but also because households become more generous. Loan supply of the wealthier expands more than proportionally to the increase of income. The opposite is true for the poor, so that the net effect of increasing inequality, by causing a savings expansion of the rich which prevails over a savings contraction of the poor, results in a net aggregate savings expansion, and a reduction of the real interest rate. This mechanism is then endogenized by replacing bequests in household's preference function by the expected present value of children gross life-time income which includes the bequest received from their parents. We first consider a single agent income type to show that the marginal propensity to save is lower when parents expect their children to be wealthier than them, and higher otherwise. So that an intergenerational increase of income inequality, when parents expect children to be relatively poorer, expands aggregate savings and lowers real interest rates. The same is valid for intra-generational income inequality. Poorer households expect their descendants to be wealthier than them, so that their bequest motive endogenously becomes inoperative; the opposite is true for the wealthier. So the poor and the rich have respectively lower and higher marginal savings rate due to operative and non-operative bequest motives, so that the net effect of increasing income inequality is a net expansion of loans supply leading to a reduction of the real interest rate as well. We describe in appendix a similar result by considering the utility of consumption of children in the parent's preference function.

Finally in section 5 we calibrate a model with endogenous output and capital to estimate how much of the real interest rates reduction in recent years can be explained by the observed increase of income inequality in US, by using our model.

3.2 Inequality, Marginal Borrowing/Saving rates, Real Interest Rates

Imagine a closed economy, in the spirit of Eggertsson and Mehrotra [21], where households live for three periods, are borrowers in the first, savers in the second, and retired in the third. During the first period of their lives they consume by borrowing from the middle age. During the second period they receive an endowment income y , and eventually a bequest from their parents, they pay-back their loans to the retired, and save for retirement by lending to the young. In the third period they retire and use their savings to consume, and to possibly leave a bequest to their children. Middle age endowment is distributed according to the density function $f(y, \bar{y}, \sigma_y)$, where \bar{y} and σ_y are respectively the mean and standard deviation of households' income. σ_y is here a measure of inequality, and \bar{y} is assumed constant.

The real interest rate in this model is given by the equilibrium in the loan market such that aggregate demand equals aggregate supply of loans at any time t . At time t young households borrow $B_t^y(r_t, y_{t+1})$ to consume. Their borrowing level is a function of the real interest rate and of their expected income during the second period of their lives y_{t+1} , that we assume is known by lenders. In addition at time t , middle aged households save $-B_t^m(r_t, y_t)$ to be able to consume when old.

Let loan demand and supply per middle age household at time t be given by the following general expressions, which depend on the real interest rate and on the middle age endowment distribution:

$$L_t^d(r_t, \bar{y}_{t+1}, \sigma_{y_{t+1}}) = \int B_t^y(r_t, y_{t+1}) f(y_{t+1}, \bar{y}_{t+1}, \sigma_{y_{t+1}}) dy_{t+1} \quad (3.2.1)$$

$$L_t^s(r_t, \bar{y}_t, \sigma_{y_t}) = - \int B_t^m(r_t, y_t) f(y_t, \bar{y}_t, \sigma_{y_t}) dy_t \quad (3.2.2)$$

The equilibrium real interest rate r_t is a solution to loan market equilibrium such

that:

$$L_t^e(r_t, \bar{y}_t, \sigma_{y_t}, \bar{y}_{t+1}, \sigma_{y_{t+1}}) = L_t^s(r_t, \bar{y}_t, \sigma_{y_t}) - L_t^d(r_t, \bar{y}_{t+1}, \sigma_{y_{t+1}}) = 0. \quad (3.2.3)$$

where we define L_t^e as excess savings at time t , and from where we derive the following equation in steady state by using the *Implicit Function Theorem*:

$$\frac{\partial L^e}{\partial r} dr + \frac{\partial L^e}{\partial \bar{y}} d\bar{y} + \frac{\partial L^e}{\partial \sigma_y} d\sigma_y = 0 \stackrel{d\bar{y}=0}{\Leftrightarrow} \frac{dr}{d\sigma_y} = -\frac{\frac{\partial L^e}{\partial \sigma_y}}{\frac{\partial L^e}{\partial r}} \quad (3.2.4)$$

If we assume that the denominator of the last expression $\frac{\partial L^e}{\partial r}$ is positive ¹, then the real interest rate decreases with increasing income inequality if $\frac{\partial L^e}{\partial \sigma_y}$ is positive. For which it is sufficient that aggregate savings increases, and aggregate borrowing decreases with increasing income inequality:

$$\frac{\partial L^s}{\partial \sigma_y} > 0 \text{ and } \frac{\partial L^d}{\partial \sigma_y} < 0 \Rightarrow \frac{\partial L^e}{\partial \sigma_y} > 0 \Rightarrow \frac{\partial r}{\partial \sigma_y} < 0 \quad (3.2.5)$$

We now split household population in two sets of relative constant sizes. A set containing the low income types with relative size equal to $\eta = \int_l f(r_t, \bar{y}_t, \sigma_{y_t}) dy_t$, and a set containing the high income types with size equal to $1 - \eta$. We assume that $\bar{y} = \eta \bar{y}_t^l + (1 - \eta) \bar{y}_t^h$ is constant over time, where $\bar{y}_t^h = \frac{1}{1-\eta} \int_h y_t f(r_t, \bar{y}_t, \sigma_{y_t}) dy_t$ is, naturally, an increasing function of σ_{y_t} .

We can also express loan supply and demand as the weighted average of loan supply and demand of low and high types. Loan supply of high types is given by $L_t^{s,h} = \frac{1}{1-\eta} \int_h L_t^s(r_t, y_t) f(r_t, \bar{y}_t, \sigma_{y_t}) dy_t$, with similar expressions for $L_t^{s,l}$, $L_t^{d,h}$ and $L_t^{d,l}$. Note

¹In chapter 1 we show that to $\frac{\partial L^s}{\partial \sigma_y} > 0 \Leftrightarrow \frac{\partial L^s}{\partial r} > \frac{\partial L^d}{\partial r}$ can be a reasonable assumption.

that²:

$$\frac{\partial L^s}{\partial \bar{y}^h} = \eta \frac{\partial L^{s,l}}{\partial \bar{y}^h} + (1 - \eta) \frac{\partial L^{s,h}}{\partial \bar{y}^h} = (1 - \eta) \left(\frac{\partial L^{s,h}}{\partial \bar{y}^h} - \frac{\partial L^{s,l}}{\partial \bar{y}^l} \right) = (1 - \eta) (MSR^h - MSR^l) \quad (3.2.6)$$

$$\frac{\partial L^d}{\partial \bar{y}^h} = \eta \frac{\partial L^{d,l}}{\partial \bar{y}^h} + (1 - \eta) \frac{\partial L^{d,h}}{\partial \bar{y}^h} = (1 - \eta) \left(\frac{\partial L^{d,h}}{\partial \bar{y}^h} - \frac{\partial L^{d,l}}{\partial \bar{y}^l} \right) = (1 - \eta) (MBR^h - MBR^l) \quad (3.2.7)$$

where $MBR^\theta \equiv \frac{\partial L^{d,\theta}}{\partial \bar{y}^\theta}$ and $MSR^\theta \equiv \frac{\partial L^{s,\theta}}{\partial \bar{y}^\theta}$ are respectively the marginal aggregate borrowing and saving rates with respect to average income of a given population segment θ .

The impact of increasing income inequality on the real interest rate is given by the sign of the marginal change of excess savings with respect to the standard deviation of endowments, which can be expressed in terms of marginal saving and borrowing rates in the following way:

$$\frac{\partial L^e}{\partial \sigma_y} = \frac{\partial L^e}{\partial \bar{y}^h} \frac{\partial \bar{y}^h}{\partial \sigma_y} = (1 - \eta) (\Delta^{hl} MSR - \Delta^{hl} MBR) \frac{\partial \bar{y}^h}{\partial \sigma_y} \quad (3.2.8)$$

Where $\Delta^{hl} MSR = MSR^h - MSR^l$. Then, given that $\frac{\partial \bar{y}^h}{\partial \sigma_y} > 0$, and by assuming that $\frac{\partial L^e}{\partial r} > 0$, the real interest rate changes with income inequality according to the following equation:

$$\frac{\partial r}{\partial \sigma_y} < 0 \Leftrightarrow \Delta^{hl} MSR > \Delta^{hl} MBR \quad (3.2.9)$$

A sufficient condition for a decreasing real interest rate with increasing income inequality is that MSR increases and MBR decreases with income, as has been evidenced in recent economic literature. In particular, Mian and Sufi [32] found recently by examining the effects of rising U.S. house prices on borrowing and spending from

² $d\bar{y}_t^h = -\frac{\eta}{1-\eta} d\bar{y}_t^l$

2002 to 2006, that on average the marginal propensity to borrow MPB increases \$0.19 per dollar of home value increase, but \$0.26 for the poorer households, and close to \$0 for the richer. Agarwal et al. [1] also found empirical evidence that the marginal propensity to borrow declines with credit scores, In their recent NBER working paper. In the savings side, besides empirically validating the fact that *the rich save more*, Dynan et al. [19] found evidence that the marginal propensity to save is higher for high income households: In income quintiles 1 and 2 they estimate that MPS increases \$0.030 for each dollar of income increase, against \$0.429 for income quintiles 4 and 5.

If empirical evidence suggests that marginal propensity to save is a positive function of income, $\Delta^{hl}MPS > 0$, and the marginal propensity to borrow is a negative function of income, $\Delta^{hl}MPB < 0$, and no borrowing or savings constraints were considered in our economy, then we could assume that $MSR = MPS$ and $MBR = MPB$ which would immediately imply by equation (3.2.9) that real interest rates would decrease with an increase of income inequality. We present next a simple model based on Eggertsson and Mehrotra [21] to inspect those mechanisms.

3.3 *Secular Stagnation* Endowment Economy Model with Bequests

We now formalize the framework described in the previous section in a specific model. We start by assuming that households have the same income type. More concretely a representative household of a generation born at time t has, for now,

the following utility function³:

$$\max_{C_t^y, C_{t+1}^m, C_{t+2}^o, Q_{t+2}} \mathbb{E}_t \left\{ \log(C_t^y) + \beta \log(C_{t+1}^m) + \beta^2 \left[\log(C_{t+2}^o) + \frac{\log(Q_{t+2})}{1 + \phi} \right] \right\} \quad (3.3.1)$$

Where C_t^y , C_t^m , and C_t^o are respectively the consumption of young, middle aged, and old; Q_t is a bequest transfered from old households to the next generation when middle aged, and $\phi \in]-1; +\infty[$ accounts for the fact that households when old may discount bequests differently than their consumption. For example if $\phi \rightarrow \infty$ parents are selfish in the sense they prefer an additional unit of old age consumption to any quantity of bequest left to children. Household budget constraints are then given by:

$$C_t^y = B_t^y \quad (3.3.2)$$

$$C_{t+1}^m = Y_{t+1}^m - (1 + r_t)B_t^y + B_{t+1}^m + \frac{Q_{t+1}}{1 + g_t} \quad (3.3.3)$$

$$C_{t+2}^o = -(1 + r_{t+1})B_{t+1}^m - Q_{t+2} \quad (3.3.4)$$

$$(1 + r_t)B_t^y \leq D_t, \text{ where } D_t = \theta Y_{t+1}^m \mu \quad (3.3.5)$$

$$0 \leq Q_t \quad (3.3.6)$$

B_t is a one period risk-free bond at an interest rate r_t . Consumption of the young is constrained by the amount they can borrow (3.3.2). The budget constraint of the middle aged is given by equation (3.3.3); they receive an income Y^m , pay their loans with interest $(1 + r)B^y$, save for retirement B^m , and receive from previous generation a bequest Q . Equation (3.3.4) is the budget constraint of old households

³For now we derive the model with a log utility function because it improves the tractability of algebraic expressions, without loss of generality, as we show in the last section and in appendix A of this chapter, where we use a reasonable calibrated CRRA preference function and introduce endogenous output and capital in the model. We assume, as in the first chapter of this dissertation, that the slope of loan supply, if negative, would never be lower in absolute terms than the slope of loan demand, meaning that the elasticity of intertemporal substitution is assumed to be greater than a minimum threshold lower than one, and also lower than standard literature EIS levels[27]. A negative loan supply slope would mean that an income effect would prevail over the substitution effect given an increase of the real interest rate.

that use their savings with interest $(1 + r)B^m$ to consume, and transfer a positive bequest Q to the next generation. Inequality (3.3.6) correspond to the assumption that bequests are positive.

We assume for now that all households are credit constrained (3.3.5), and that the borrowing limit of the young is binding. Later we relax this assumption. We also assume that the borrowing limit is a positive function of expected lifetime income ($\mu > 0$ and $\theta > 0$), and known by the lenders when agents are young.

The equilibrium real interest rate of this model solves loan market equilibrium equation, requiring that the demand for loans L_t^d equals supply L_t^s at any time t , or that $N_t^y B_t^y$ is equal to $-N_t^m B_t^m$, equivalent to:

$$L_t^d = (1 + g_t)B_t^y = -B_t^m = L_t^s \quad (3.3.7)$$

where $N_t^m = N_{t-1}^y$ and $1 + g_t = \frac{N_t^y}{N_{t-1}^y}$ is defined as the growth rate of births.

Loan Demand

From equation (3.3.2) and inequality (3.3.5) corresponding to the binding borrowing limit, we derive an expression for the consumption of the young given by:

$$C_t^y = \frac{D_t}{1 + r_t} \quad (3.3.8)$$

Using inequality (3.3.5) and assuming the borrowing limit is binding, loan demand is given by:

$$L_t^d = (1 + g_t)B_t^y = \frac{1 + g_t}{1 + r_t} D_t \quad (3.3.9)$$

In order to model D_t as a binding borrowing limit for all households we need to ensure that the constrained demand for loans L_t^d is lower than the unconstrained

$L_t^{d,u}$, which is a linear positive function of income:

$$L_t^d = \frac{1+g_t}{1+r_t} D_t \leq \frac{1+g_t}{1+r_t} \frac{Y_{t+1}^m}{1+\beta+\beta^2} = L_t^{d,u} \Leftrightarrow D_t = \theta Y_{t+1}^m \mu \leq \frac{Y_{t+1}^m}{1+\beta+\beta^2} \quad (3.3.10)$$

By Setting $\mu \in]0; 1[$ and $\theta < \frac{Y_{min}^{1-\mu}}{1+\beta+\beta^2}$ we ensure that the limit is binding for all values of $Y \in]Y_{min}; Y_{max}[$. Those parameters also ensure that the marginal borrowing rate MBR is a negative function of income, consistent with what is empirically expected for the marginal propensity to borrow:

$$MBR = \frac{\partial L_t^d}{\partial Y_{t+1}} = \frac{1+g_t}{1+r_t} \theta \mu Y_{t+1}^{\mu-1}, \text{ and } \frac{\partial MBR}{\partial Y_{t+1}} < 0 \quad (3.3.11)$$

Later we endogenize the fact that $\frac{\partial MBR}{\partial Y} < 0$ by questioning the assumption $\mu \in]0; 1[$ and test an alternative where $\mu \geq 1$, and the binding borrowing limit assumption relaxed for the wealthier.

Loan Supply

The middle aged are at an interior solution and satisfy a consumption Euler equation given by:

$$\beta \mathbb{E}_t \frac{C_t^m}{C_{t+1}^o} = \frac{1}{1+r_t} \quad (3.3.12)$$

Using equation (3.3.12) in the budget constraint of the old (3.3.4)⁴, we get an expression for the consumption of the middle aged given by:

$$C_t^m = \frac{C_{t+1}^o}{\beta(1+r_t)} = \frac{-(1+r_t)B_t^m - Q_{t+1}}{\beta(1+r_t)} = -\frac{B_t^m}{\beta} - \frac{Q_{t+1}}{\beta(1+r_t)} \quad (3.3.13)$$

By combining this expression for C_t^m with the middle aged budget constraint (3.3.3) we derive an expression for Loan supply with bequests, which is greater or equal

⁴as well as the deterministic nature of the model.

than the corresponding expression with an inoperative bequest motive ($Q = 0$):

$$L_t^s = -B_t^m = \frac{\beta}{1+\beta}(Y_t^m - D_{t-1}) + \frac{\beta}{1+\beta} \left(\frac{Q_t}{1+g_t} + \frac{Q_{t+1}}{\beta(1+r_t)} \right) \quad (3.3.14)$$

Furthermore, the marginal savings rate MSR with an inoperative bequest motive is a positive function of income:

$$MSR = \frac{\partial L_t^s}{\partial Y_t} = \frac{\beta}{1+\beta} (1 - \theta\mu Y_t^{\mu-1}), \text{ and } \frac{\partial MSR}{\partial Y_t} > 0 \quad (3.3.15)$$

Note that the concavity of the marginal borrowing rate with respect to income is here explaining the convexity of the marginal savings rate with respect to income, which is itself the consequence of an exogenous parametrization of the borrowing limit given by assuming that $\mu < 1$. We later relax this assumption together with not requiring a binding borrowing limit for the wealthier households, and get a similar result. Furthermore we will analyze a mechanism only dependent on the loan supply side of the model based on bequests.

Bequests

We now inspect how the level of generosity of parents towards children affect the marginal savings rate and the natural rate of interest. In particular we compare expressions with and without an operative bequest motive. By combining the First Order Conditions for C_{t+2}^o and Q_{t+2} with the middle age budget constraint we derive the following expression for expected bequest of next period⁵:

$$\mathbb{E}_t Q_{t+1} = (1+r_t)\Psi \left(Y_t^m - D_{t-1} + \frac{Q_t}{1+g_{t-1}} \right) > 0 \quad (3.3.16)$$

The constant $\Psi = \frac{\beta}{\beta+(1+\phi)(1+\beta)} \in]0; 1[$ for $\phi \in]-1; +\infty[$ can be interpreted as a generosity coefficient of parents towards children. The bequest to descendants at time $t+1$ is expected to be higher at time t for a higher interest rate at time t .

⁵The endowment economy model with a warm glow bequest motive is derived in Appendix A.

Moreover, with this preference function the bequest motive is always strictly positive and operative unless $\phi = +\infty \Rightarrow \Psi = 0$. This becomes evident when loan market is in equilibrium and the real interest rate is the natural rate of interest r_t^n :

$$\mathbb{E}_t Q_{t+1}^n = \frac{1 + g_t}{2 + \phi} D_t \quad (3.3.17)$$

By replacing the bequest expression (3.3.16) in the loan supply expression (3.3.14), we derive a general expression for loan supply depending only on bequest received by the previous generation during middle age, that we will use through the rest of the chapter⁶:

$$L_t^s = \frac{\beta + \Psi}{1 + \beta} \left(Y_t^m - D_{t-1} + \frac{Q_t}{1 + g_{t-1}} \right) \quad (3.3.18)$$

This is a useful expression in particular when we assume later on that children and parents income types are *iid*, so that the average expected bequest received by any income type is the same. In this single income type framework, we now derive the marginal savings rate when loan market is in equilibrium, from loan supply equilibrium, given by the following expression:

$$L_t^s = \frac{\beta + \Psi}{1 + \beta} \left[Y_t^m - \frac{1 + \phi}{2 + \phi} D_{t-1} \right] > L_t^{s, Q=0} = \frac{\beta}{1 + \beta} [Y_t^m - D_{t-1}] \quad (3.3.19)$$

As expected, loan supply in equilibrium expands with an operative bequest motive, which results in a lower natural rate of interest:

$$1 + r_t^n = \left(\frac{1 + \beta}{\beta + \Psi} \right) \frac{(1 + g_t) D_t}{Y_t^m - \frac{1 + \phi}{2 + \phi} D_{t-1}} < 1 + r_t^{n, Q=0} = \frac{1 + \beta}{\beta} \frac{(1 + g_t) D_t}{Y_t^m - D_{t-1}} \quad (3.3.20)$$

⁶As noted already, loan supply with a log utility function is inelastic with respect to the real interest rate, meaning that income and substitution effects cancel each-other. This makes algebraic expressions more tractable, without loss of generality, since for reasonable low EIS values (lower than one), when income effect prevails over the substitution effect, the negative slope of loan supply with respect to the real interest rate does not change the impact sign on the real interest rate of a loan supply expansion.

Regarding the marginal savings rate given by expression below we can observe that it increases when the bequest motive is operative, and also increases with income.⁷:

$$MSR_t = \frac{\beta + \Psi}{1 + \beta} \left[1 - \frac{1 + \phi}{2 + \phi} D'_{t-1} \right] > \frac{\beta}{1 + \beta} \left[1 - D'_{t-1} \right] = MSR_t^{Q=0} \quad (3.3.21)$$

$$MSR'_t = \frac{\beta + \Psi}{1 + \beta} \frac{1 + \phi}{2 + \phi} (-D''_{t-1}) > 0 \quad (3.3.22)$$

But we can also observe that the convexity of MSR with respect to income is a direct consequence of the concavity of the marginal borrowing rate. Consequently with this preference function, an increase of income inequality would impact negatively the natural rate of interest from the savings side, only as long as the marginal borrowing rate is negative sloped with respect to income. Unless generosity with respect to the next generation increases when agents become wealthier. We explore those mechanisms in the next section by introducing income heterogeneity in the model, and inspect separately the impact of loans demand and supply sides on the natural rate of interest when inequality increases, respectively by canceling the bequest motive, and the borrowing limit concavity.

3.4 Decreasing real interest rates, with increasing income inequality

Inequality is introduced in the model by considering two types of agents with different endowments levels when middle-aged. We assume that the average endowment at time t is always constant, and given by:

$$Y_t^m = \eta Y_t^{m,l} + (1 - \eta) Y_t^{m,h} = Y^m \quad (3.4.1)$$

$${}^7 D'_{t-1} = \frac{\partial D_{t-1}}{\partial Y_t} = \theta \mu Y_t^{\mu-1} > 0, D''_{t-1} = \theta \mu (\mu - 1) Y_t^{\mu-2} < 0, \text{ and } MSR'_t = \frac{\partial MSR_t}{\partial Y_t}.$$

Table 3.1: Summary of model conditions for increasing inequality to decrease r

General condition for $\frac{\partial r}{\partial \sigma_y} < 0 : \Delta^{hl}MSR > \Delta^{hl}MBR$		
Sufficient conditions for $\frac{\partial r}{\partial \sigma_y} < 0$:	$\Delta^{hl}MBR < 0$	$\Delta^{hl}MSR > 0$
Borrowing mechanism	✓	✓
- $\mu < 1$, and borrowing limit binding for all agents:	✓	✓
- or $\mu > 1$, and borrowing limit binding just for the poor:	✓	✓
Savings mechanism		✓
- $\phi'(y) < 0$: Generosity increases exogenously with income:		✓
- or bequest ZLB, $Q \geq 0$, becomes binding for the poor:		✓

η is the fraction of low income households. The standard deviation of this income distribution at time t and its derivative with respect to high type income are respectively given by the following two expressions:

$$\sigma_{y_t} = (Y_t^{m,h} - Y^m) \sqrt{\frac{1-\eta}{\eta}}, \text{ and } \frac{d\sigma_{y_t}}{dY_t^{m,h}} = \sqrt{\frac{1-\eta}{\eta}} \quad (3.4.2)$$

The natural rate of interest changes with an increase of income inequality according to:

$$\frac{\partial r}{\partial \sigma_y} = - \left(\frac{\sqrt{\eta(1-\eta)}}{\frac{\partial L^e}{\partial r}} \right) (\Delta MSR_t^{hl} - \Delta MBR_t^{hl}) \quad (3.4.3)$$

So the equilibrium real interest rate decreases with increasing income inequality if $\Delta MSR_t^{hl} > \Delta MBR_t^{hl}$. We now inspect separately the mechanisms respectively related to the borrowing and supply sides of the model, briefly summarized in table(3.1).

3.4.1 Borrowing constraints, inequality, and real interest rates

We cancel the bequest motive in this subsection by setting $\phi = +\infty$. From expressions (3.3.11) and (3.3.21) we derive ΔMSR_t^{hl} and MBR_t^{hl} and observe that the condition is verified, so that the real interest rate in this model decreases with an

increase of income inequality:

$$\Delta MBR_t^{hl} = \frac{1+g_t}{1+r_t}(D_t^{h'} - D_t^{l'}) < 0 \quad (3.4.4)$$

$$\Delta MSR_t^{hl} = -\frac{\beta}{1+\beta}(D_t^{h'} - D_t^{l'}) > 0 \quad (3.4.5)$$

Note that $\mu < 1$ and $D_t^{h'} = \theta\mu Y_t^{h\mu-1} < \theta\mu Y_t^{l\mu-1} = D_t^{l'}$. It is the concavity of the borrowing limit that determines the above result, of an increase of income inequality lowering the equilibrium real interest rate. In fact, if $\mu = 0$ as in Eggertsson and Mehrotra [21], or $\mu = 1$ so that the borrowing limit is a linear function of expected income, then $\Delta MBR_t^{hl} = \Delta MSR_t^{hl} = 0$ and an increase of income inequality would not impact the real interest from the borrowing side of this economy.

i) Assumption that all agents are credit constrained when young

If we assume that all agents are borrowing constrained when young, it is also reasonable to assume the concavity of the borrowing limit, consistently with their marginal propensity to borrow, and in contrast with a constant borrowing limit presuming that lenders have no information whatsoever about borrowers income type, or, in the other extreme, in contrast with a linear function of expected income that would presume lenders have full information for a given lending motive.

ii) Assumption that only the low income type are credit constrained

We now recall the work of Agarwal et al. [1] who showed that credit card limits in US not only increase with credit scores, but so does the marginal propensity to lend from banks. The borrowing limit in our model would be an increasing convex function with respect to expected income, with $\mu > 1$, but would only bind for the low income types. We further assume that for average income \bar{Y}^m constrained and

unconstrained borrowing levels would be equal:

$$Y \leq \bar{Y} : L_t^d(Y) = \frac{1 + g_t}{1 + r_t} \theta Y^\mu \quad (3.4.6)$$

$$Y \geq \bar{Y} : L_t^{d,u}(Y) = \frac{1 + g_t}{1 + r_t} \frac{Y}{1 + \beta + \beta^2} \quad (3.4.7)$$

$$Y = \bar{Y} : L_t^d(\bar{Y}) = L_t^{d,u}(\bar{Y}) \Leftrightarrow \theta = (1 + \beta + \beta^2)^{-1} \quad (3.4.8)$$

Without loss of generality we assume that $\bar{Y} = 1$. Then for the borrowing limit to be binding for the low income type and non binding for the higher, we need $\mu > 1$:

$$L_t^{d,u}(Y_{t+1}^l) < L_t^d(Y_{t+1}^l) \Leftrightarrow \frac{1 + g_t}{1 + r_t} \theta (Y_{t+1}^l)^\mu < \frac{1 + g_t}{1 + r_t} \theta Y_{t+1}^l \Leftrightarrow (Y_{t+1}^l)^{\mu-1} < 1 \quad (3.4.9)$$

$$L_t^{d,u}(Y_{t+1}^h) > L_t^d(Y_{t+1}^h) \Leftrightarrow \frac{1 + g_t}{1 + r_t} \theta (Y_{t+1}^h)^\mu > \frac{1 + g_t}{1 + r_t} \theta Y_{t+1}^h \Leftrightarrow (Y_{t+1}^h)^{\mu-1} > 1 \quad (3.4.10)$$

From where the marginal borrowing rates of high and low types would be given by:

$$MBR_t^l = \frac{1 + g_t}{1 + r_t} \theta \mu Y_{t+1}^{l \mu-1} \quad (3.4.11)$$

$$MBR_t^h = \frac{1 + g_t}{1 + r_t} \theta \quad (3.4.12)$$

The marginal borrowing rate of the low types is higher than the high types for reasonable values of μ and Y^l given by the condition $Y^l > h(\mu) = \left(\frac{1}{\mu}\right)^{\frac{1}{\mu-1}}$, where $h(\mu)$ is a positive function of μ ⁸. In that case:

$$\Delta MBR_t^{hl} = -\frac{1 + g_t}{1 + r_t} \theta \left(\mu Y_{t+1}^{l \mu-1} - 1 \right) < 0 \quad (3.4.13)$$

$$\Delta MSR_t^{hl} = \frac{\beta}{1 + \beta} \theta \left(\mu Y_{t+1}^{l \mu-1} - 1 \right) > 0 \quad (3.4.14)$$

Having inspected the impact of borrowing limits on inequality/real interest rate

⁸For reference $h(1^+) = 0.37$ and $h(4) = 0.63$

dynamics, we now look at the savings side of the model with an operative bequest motive.

3.4.2 Bequests, inequality, and real interest rates

In their paper "*Do the Rich Save More*" Dynan et al. [19] show that higher savings rates are associated to higher lifetime income. They also find evidence that the *marginal savings rate* (MSR) is a positive function of lifetime income. The results of previous sections are consistent with their findings, in particular due to the fact that the marginal savings rate of the rich are higher because their marginal borrowing rate are lower. Dynan et al. [19] inspect several factors explaining an increasing marginal savings rate with respect to income, that would result in the reduction of the natural rate of interest when income inequality increases. We next analyze the conditions under which a bequest motive is one of those factors. In order to do so, we cancel the concavity of the borrowing limit by setting $\mu = 0$ so that the marginal borrowing rate $MBR = 0$ as in Eggertsson and Mehrotra [21]. We also assume that the borrowing limit is binding for all households. Aggregate loan supply for each income type is given by:

$$L_t^{s,i} = \frac{\beta + \Psi}{1 + \beta} \left(Y_t^{m,i} - D_{t-1} + \frac{Q_t}{1 + g_{t-1}} \right) \quad (3.4.15)$$

We assume through the rest of the chapter that parents and children income types are *iid* so that on average the bequest received from parents is the same among income types, and has an average steady state expression independent of income types:

$$Q = \frac{Y^m - D}{\frac{1}{\Psi(1+r)} - \frac{1}{1+g}} \quad (3.4.16)$$

Note that average bequest naturally increase with generosity, and decrease with population growth since more children mean a lower parcel of the same bequest per child. Then in the present model the marginal savings rate is independent of income types, thus not triggering the expected mechanism:

$$MRS_t^i = \frac{\beta + \Psi}{1 + \beta} \Rightarrow \Delta MSR_t^{hl} = 0 \quad (3.4.17)$$

An increase of inequality expands savings of the high type, and contracts savings of the low type. Because the marginal savings rate of high and low types are equal, the two effects cancel each other not affecting aggregate loan supply and equilibrium real interest rate. This model as is does not capture the fact that the poor may have less propensity to leave a bequest to their children because they prioritize consumption, or may expect their children to be better off. A straightforward way to account for this is to assume that the level of generosity of households with respect to the next generation increases exogenously with income.

i) Generosity increases exogenously with lifetime income

As seen earlier, the parameter ϕ is a measure of selfishness. When ϕ tends to infinity bequests tend to zero, as well as Ψ . The parameter Ψ can then be interpreted as a measure of generosity towards the next generation. We now assume that the level of generosity increases with income $Y_t^{m,i}$. Let $\Psi^i = \Psi(Y_t^{m,i}) = \psi Y_t^{m,i}$, where ψ is a positive constant. This is equivalent to set $\phi = \phi(Y_t^{m,i}) = \frac{\beta}{1+\beta} \left(\frac{1}{\psi Y_t^{m,i}} - 1 \right)$ which, being a measure of selfishness, is now a decreasing function of lifetime income. The supply of loans for a given income type is given by:

$$L_t^{s,i} = \frac{\beta + \psi Y_t^{m,i}}{1 + \beta} \left(Y_t^{m,i} - D_{t-1} + \frac{Q_t}{1 + g_{t-1}} \right) \quad (3.4.18)$$

Now the steady state average bequest expression increases with increasing income

inequality:

$$Q = (1+r) \frac{\psi(Y^h - Y^l) - \Psi D}{1 - \Psi \frac{1+r}{1+g}} = \frac{1}{\frac{1}{\Psi(1+r)} - \frac{1}{1+g}} \left[\frac{Y^h - \bar{Y}}{\eta} - D \right] \quad (3.4.19)$$

If the rich are more generous towards their descendants then the bequest increase of the wealthier prevails over the bequest contraction of the poorer when inequality increases. The same result is obtained for the marginal savings rate. Then $\Delta MSR_t^{hl} > 0$ ⁹:

$$\Delta MSR_t^{hl} = \frac{1}{1+\beta} \left[2(\Psi^h - \Psi^l) + \frac{\beta + \Psi^h}{1+g} \frac{\partial Q}{\partial Y^h} - \frac{\beta + \Psi^l}{1+g} \frac{\partial Q}{\partial Y^l} \right] > 0 \quad (3.4.20)$$

The fact that rich and poor discount bequests differently is sufficient to trigger the mechanism by which an increase of income inequality has a negative effect on the equilibrium real interest rate level. Next we endogenize this mechanism by making an adjustment of the bequest motive in the preference function, such that households leave bequests because they compare expected wealth of their descendants with themselves, and are willing to help if they expect children to be poorer, but not if they expect them to be wealthier.

ii) Generosity increases endogenously with lifetime income

We now assume that agents trade-off consumption by expected gross wealth of next generation, instead of bequests directly. They care for their children expected wealth relative to their own, in contrast of just caring for leaving a bequest to the next generation, independently of whether they need their help or not. By next generation expected gross wealth we mean the sum of their expected endowment and the bequest received from their parents. In this case the bequest zero lower bound given by (3.3.6) may become binding. The utility function of a representative

⁹Note that $\frac{\partial Q}{\partial Y^h} > 0$ and $\frac{\partial Q}{\partial Y^l} < 0$. Furthermore $\Psi = \psi Y^m$, $\Psi^h = \psi Y^{m,h}$ and $\Psi^l = \psi Y^{m,l}$.

household born at time t becomes:

$$\max_{C_t^y, C_{t+1}^m, C_{t+2}^o, W_{t+2}^m} \mathbb{E}_t \left\{ \log(C_t^y) + \beta \log(C_{t+1}^m) + \beta^2 \left[\log(C_{t+2}^o) + \frac{\log(W_{t+2}^m)}{1 + \phi} \right] \right\} \quad (3.4.21)$$

where $W_t^m = Y_t^m + \frac{Q_t}{1+g_{t-1}}$, and the utility is maximized subject to the same budget constraints (3.3.2),(3.3.3),(3.3.4) and inequalities (3.3.5) and (3.3.6). We use the same methodology as in previous sections to derive an expression for the bequest transferred to the next generation, when bequest zero lower bound is not binding:

$$Q_{t+1} = (1 + r_t)\Psi_t(W_t^m - D_{t-1}) - (1 - \Psi_t)(1 + g_t)\mathbb{E}_t Y_{t+1}^m \quad (3.4.22)$$

where $\Psi_t = \frac{\beta}{\beta + \frac{(1+\phi)(1+\beta)}{1+g_t}} \in]0; 1[$ for $\phi \in]-1; +\infty[$. The previous expression for Q_{t+1} may become negative if descendants endowment present value is expected to be higher than a threshold depending on the net wealth and generosity of their parents:

$$Q_{t+1} \geq 0 \Leftrightarrow \frac{\beta}{1 + \beta} \frac{W_t^m - D_{t-1}}{1 + \phi} \geq \frac{\mathbb{E}_t Y_{t+1}^m}{1 + r_t} \quad (3.4.23)$$

Loan supply is conditional on the bequest being binding and can be expressed by:

$$L_t^s = \frac{\beta}{1 + \beta} (W_t^m - D_{t-1}) + \frac{(1 - \Psi_t)(1 + g_t)}{1 + \beta} \max \left[0, \frac{\beta}{1 + \beta} \frac{W_t^m - D_{t-1}}{1 + \phi} - \frac{\mathbb{E}_t Y_{t+1}^m}{1 + r_t} \right] \quad (3.4.24)$$

In this economy, if the present value of children expected endowment is sufficiently low compared to the net wealth of their parents, and considering their level of generosity, then the bequest motive becomes operative and savings expand with respect to an inoperative bequest motive state. The expansion of parents savings increases further with expectations that children are poorer. Then the marginal savings rate when the bequest motive is operative is greater higher than when it is

not:

$$MSR_t^Q = \frac{\beta + \Psi_t}{1 + \beta} > \frac{\beta}{1 + \beta} = MSR_t \quad (3.4.25)$$

This is a relevant expression when intra-generation inequality is introduced in the model, because wealthier agents tend to expect their children to be relative less wealthy, and then be more generous, in contrast with poorer households that expect their children to be better off than them, and then tend to be less generous. But before that we will still present the following expression for the natural rate of interest when the bequest motive is operative:

$$1 + r_t = \frac{1 + \beta}{\beta + \Psi_t} \frac{(1 + g_t) \left[D_t + \frac{1 - \Psi_t}{1 + \beta} Y_{t+1}^m \right]}{Y_t^m - D_{t-1} + \frac{Q_t}{1 + g_{t-1}}} < 1 + r_t^n \quad (3.4.26)$$

Then, when the bequest motive is operative the natural rate of interest decreases if generosity towards children Ψ_t increases, or if expected income of descendants decreases. The natural rate of interest may increase until an upper bound given by its expression with an inoperative bequest motive, when children expected endowment increase above a given threshold.

Note that, an expected increase of inter-generation inequality reduces the *natural rate of interest* in this model.

If children are expected to become relatively poorer than parents, (or parents become relatively wealthier than children) then the supply of loans expands causing a reduction of the equilibrium real interest rate.

We now assume that there is intra-generation income inequality in our economy as in previous sections: $\eta_s Y_t^{m,l} + (1 - \eta_s) Y_t^{m,h} = Y^m$, and that children and parents types are *iid*. We further assume that parents don't know their children income type when they make the bequest decision, so that children expected income is equal to Y^m . Then poorer households expect descendants to be richer than themselves, and

the richer expect their children to be poorer. In this case the condition for a positive bequest is given by the following equation:

$$Q_{t+1}^{\gamma,i} \geq 0 \Leftrightarrow \frac{\beta}{1+\beta} \frac{W_t^{m,\gamma,i} - D_{t-1}}{1+\phi} \geq \frac{Y^m}{1+r_t} \quad (3.4.27)$$

$$\Leftrightarrow Y_t^\gamma + \frac{Q^i}{1+g_{t-1}} \geq \frac{(1+\beta)(1+\phi)}{\beta} \frac{Y^m}{1+r_t} + D_{t-1} \quad (3.4.28)$$

where $\gamma \in \{l, h\}$, and i is household's id. Note that this condition is the same for all households, since the right-hand side of the equation(3.4.28) does not depend on any income type in particular. For the subset above the threshold the marginal savings rate is equal to $\frac{\beta+\Psi_t}{1+\beta}$ and higher than the marginal savings rate of the ones in the subset below $\frac{\beta}{1+\beta}$. A sufficient condition for an operative bequest motive for all high income types is given by:

$$Y_t^h > \frac{(1+\beta)(1+\phi)}{\beta} \frac{Y^m}{1+r_t} + D_{t-1} \Rightarrow Q_{t+1}^{h,i} > 0 \quad (3.4.29)$$

If there exists at least one low type household for which the bequest zero lower bound binds then the aggregate marginal savings rate of low types is lower than the one of the high types, such that the conditions for the mechanism linking an increase of income inequality to lower interest rates are satisfied. In the next section and in appendix we show that under reasonable assumptions low types may have their bequest motive always inoperative, while always operative for high types. In that case we can derive a close form expression for the derivative of the natural rate of interest with respect to a measure of income inequality:

$$\frac{\partial r}{\partial \sigma_y} = - \left(\frac{\sqrt{\eta(1-\eta)}}{\frac{\partial L^e}{\partial r}} \right) (\Delta MSR_t^{hl} - \Delta MBR_t^{hl}) = - \left(\frac{\sqrt{\eta(1-\eta)}}{\frac{\partial L^e}{\partial r}} \right) \frac{\Psi_t}{1+\beta} < 0 \quad (3.4.30)$$

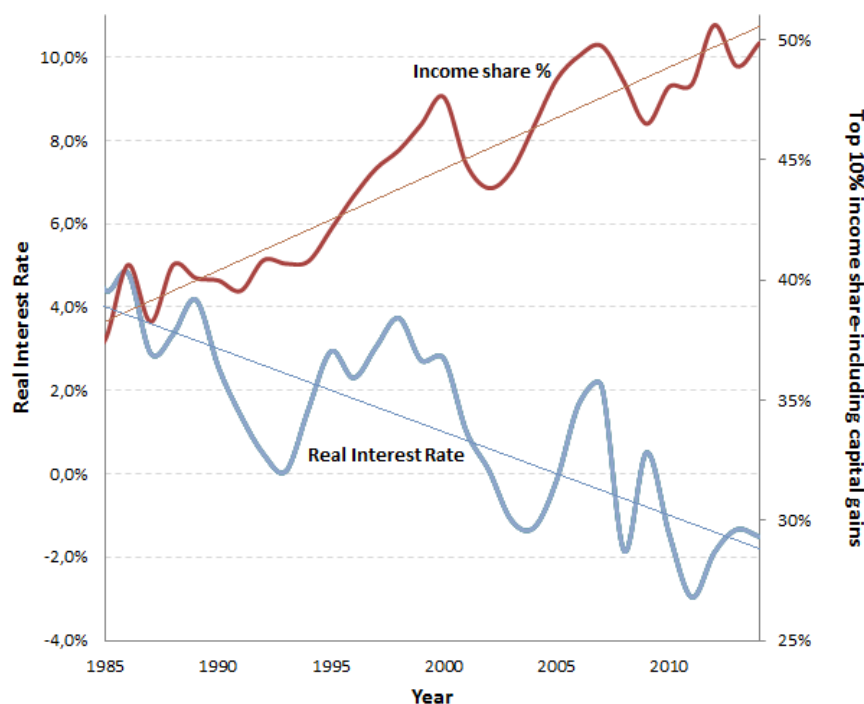
where $\frac{\partial L^e}{\partial r} = \frac{1+g_t}{(1+r_t)^2} \left((1-\eta)Y^m \frac{1-\Psi_t}{1+\beta} + D_{t-1} \right) > 0$.

3.5 Quantitative calibration of the model

In previous sections we were able to formally capture some of the mechanisms supporting the relation between increasing income inequality and decreasing real interest rates, although with a simple and stylized OLG model. We nevertheless think it would be of value to try to estimate by how much our model, explicitly parametrized, could explain the reduction of real interest rates in recent years.

In the period between 1985 and 2005 the real interest rates in US have fallen from 4.4% to -0.2% , while the share of the wealthier population decile increased 10 percentage points, from 38% to 48% (Figure: 3.3)¹⁰. How much of the real interest rate reduction of -4.6% during that period is our model able to explain, is the question we try to answer next.

Figure 3.3: Real Interest Rates and Income Inequality in US



Source: The World Wealth and Income Database - WID (Piketty); Fred

¹⁰Source: The World Wealth and Income Database - WID (Piketty); Fred

We start by verifying how much of the effective reduction of the real interest rate during the observation period is explained by a base case calibration of our model due to an increase of income inequality. We then test the robustness of the real interest rate reduction to calibration changes of relevant parameters.

The model used for calibration is the *warm glow* type version where generosity endogenously increases with agents lifetime income. The endogenous bequest motive is modeled by considering descendants expected wealth in agents preference function, rather than only the bequest itself. Expected wealth being equal to the sum of descendants expected lifetime income and bequest received from parents. It is assumed that labor endowment types of agents and their descendants are independent, so that expected income of descendants of the wealthier and poorer is the same, and equal to expected average income of all households of their generation. Agents expecting wealthier descendants than themselves have lower incentives to leave bequests, and would consequently save relatively less than wealthier agents that expect their descendants to be relatively poorer, and would then save more. We also introduce capital in the model as in Eggertsson and Mehrotra [21]¹¹, and allow for a CRRA utility function where the elasticity of inter-temporal substitution may be different than one. A steady state equilibrium for this economy is defined with the requirement that average total bequests received by middle-age household during one period is equal to the average total bequests left by the old during the next period.

The calibration of the benchmark model is described in Table 1, with usual annual parameters calibrated according to recent literature, namely the discount rate β ¹², the elasticity of inter-temporal substitution $\frac{1}{\sigma}$, depreciation rate δ ¹³, and the loan collateral concavity μ . In the model, the discount rate β and the depreciation rate

¹¹Model with capital, a *warm glow* type bequest motive, and a CRRA utility function described in Appendix A

¹² β is calibrated at 0.571 for a period of 20 years, corresponding to 0.972 for a period of one year.

¹³ δ is calibrated at 0.88 for a period of 20 years, corresponding to 0.10 for a period of one year.

Table 3.2: Parameters and Simulation Results

Description	Parameter	Value
Calibration:		
Discount rate	β	0.97
Intertemporal substitution	$\frac{1}{\sigma}$	0.5
Descendants wealth discount rate	ϕ	1.01
Collateral concavity ($D = \theta Y_i^\mu$)	μ	0.50
Depreciation rate (year)	δ	0.1
Population growth	g	0.7%
Low income type population share	η	90%
Average labor endowment	L	1
Low income type labor endowment	L^l	0.69
High income type labor endowment	L^h	3.80
Matching:		
High type population share	$1 - \eta$	10%
High type income share 1985	$\frac{(1-\eta)Y_{ini}^h}{Y}$	38%
High type income share 2005	$\frac{(1-\eta)Y_{ss}^h}{Y}$	48%
Mean initial estate size/lifetime income	Q/Y	3.6%
Debt to income ratio	D/Y	0.36
Capital to annual income ratio	k/y	3.0
Initial real interest rate	r_{ini}	4.4%
Results:		
Final real interest rate - model	r_{ss}	3.4%
real interest rate change - model	$\Delta_{mod} r$	-1.0%
real interest rate change - observed	$\Delta_{obs}^{85-05} r$	-4.6%

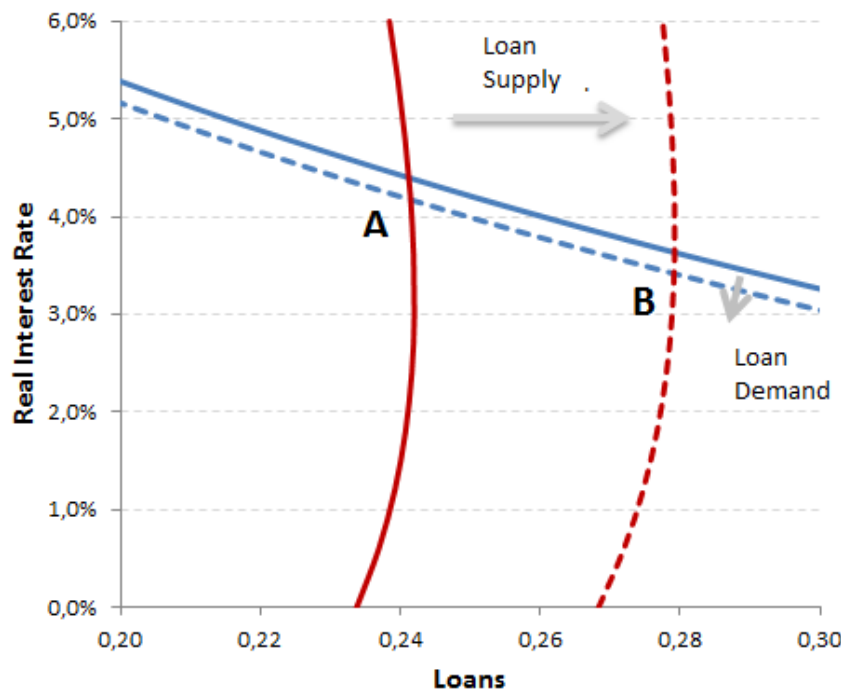
δ are adjusted for a period of 20 years. We match initial and final top decile income shares by adequately calibrating labor endowments of the wealthier and poorer at initial and final steady states. We then calibrate the discount factor β combined with the bequest discount rate (or selfish parameter ϕ), to match the initial steady state values of the real interest rate, and average estate level as a proxy of the bequest level cap (Hendricks [28]), with the observed values in the beginning of the observation period. The collateral multiplier θ , is calibrated to obtain an initial reasonable debt limit to income ratio $\frac{D}{Y}$, after setting the collateral concavity.

In an OLG model with three periods of 20 years each, the implied labor share of the Cobb-Douglas production function α_{20} , should be greater than the typical approximate value of two thirds. By setting $\alpha_{20} = 0.86 > \frac{2}{3}$, the capital to annual income ratio in the initial steady state is set to 3.0.

The benchmark model is able to explain around 22% of the effective real interest rate reduction during the observation period: An increase of the top decile income share from 38% to 48% would lead to a real interest rate reduction of 1.0%, from 4.4% to 3.4%, in contrast with an observed total reduction of 4.6%, from 4.4% to -0.2% . Low income agents do not leave bequests to their descendants in the initial and final steady states, which is consistent with recent literature (Benhabib et al. [6]). But high income types increase the level of bequests left at the end of their lives. The marginal increase in savings of high types, due to an increase of income inequality, prevails over the savings contraction of low types, resulting into a net expansion of loan supply, and a consequent reduction of the *natural rate of interest* (figure 3.4). Loan demand contracts because of the concavity of the borrowing limit.

We test the robustness of our results by changing some parameters one by one to commonly used values (Table: 2), and by checking the impact on the reduction level of the natural rate of interest, although ensuring the initial real interest rate

Figure 3.4: Equilibrium in the Loan Market



Source: The World Wealth and Income Database - WID; FRED

and average estate values match observations in the beginning of the period. The change magnitude of the natural rate of interest due to an increase of 10 percentage points of top decile income share is generally robust to other parameter changes, in particular for significantly different initial bequest to income ratio levels.

Although the increase of income inequality between 1985 and 2005 only seems to account, according to our model, for around 22% of the reduction of the real interest rate during that period, it is nevertheless not an insignificant value. There are other relevant factors that certainly may explain the difference, namely age structure changes in population, like life expectancy and the retirement age (see chapter 2), or debt deleveraging, a birth rate slow-down, or a reduction of the price of investment (Eggertsson and Mehrotra [21]). Combining all those factors in a single multi-generations model in order to better understand the relative contributions of each of those factors in a consistent way is part of our work going forward.

Table 3.3: Robustness analysis

INPUTS			OUTPUTS	
Description	parameter	benchmark	sensibility	Δr
Δ Real Int. Rate	Δr	-0.98%		-0.98%
EIS	$\frac{1}{\sigma}$	0.5	1.0	-0.64%
Depreciation (year)	δ	0.1	0.2	-1.02%
Population growth	g	0.7%	0.0%	-1.04%
Collateral multilier	θ	0.45	0.35	-1.04%
Collateral concavity	μ	0.5	0.0	-0.67%
Bequest/lifetime income	$\frac{Q}{Y}$	$3, 6\%$	$[1.0\%; 10.0\%]$	$[-0.91\%; -1.13\%]$

3.6 Final remarks

In this chapter, we formalize the relation between increasing income inequality and low real interest rates using an overlapping generations model with borrowing constraints and a bequest motive. The underlying mechanism in our model relating real interest rates and inequality is based on empirical evidence in recent literature that households' marginal borrowing and saving rates are respectively negative and positive functions of income, so that the net effect on aggregate borrowing and savings of a permanent increase of income inequality is respectively a net contraction and a net expansion, that may lead to a persistent reduction of the *natural rate of interest*.

In particular, the borrowing mechanism in our model is based on the concavity of the marginal propensity to borrow, and on binding borrowing constraints, both consistent with empirical observations in recent literature[1][32]. The saving mechanism is illustrated with an endogenous propensity of households to be more generous with respect to their children by leaving them greater bequests, when they are expected to be relatively poorer. In the opposite direction, if agents expect their direct descendants to be much wealthier than them, then the bequest motive may become inoperative. So that wealthier households are more generous, which makes their marginal savings rate higher than the poorer households that leave lower or no be-

quests at all, as also observed by Hendricks [28]. In this model the savings channel through which inequality affects the real interest rate is then the bequest motive, that is endogenously turned on or off respectively for the rich and the poor, making the marginal savings rate of the rich greater than the poor, so that an increase of inequality triggers a net expansion of aggregate savings that drags down the *natural rate of interest*.

Our main contribution in this chapter is to present an explicit formalism linking low real interest rates with increasing income inequality, by gathering and building on some relevant topics in recent literature, namely increasing income inequality[33], bequests[4][7], the question of whether higher-lifetime income levels lead to higher marginal propensity to save[19] and lower marginal propensity to borrow[1][32], and *secular stagnation*[21][34].

3.A Endogenous Output and Capital

Here we derive the model with endogenous output and capital, its equilibrium conditions, as well as operative bequest conditions for each income type. We assume that only the middle age supply labor and capital to competitive firms that take wages and rental capital rates as given, and maximize profits subject to a standard Cobb-Douglas production function:

$$Z_t = \max_{L_t, K_t} Y_t - w_t L_t - r_t^k K_t \quad (3.A.1)$$

$$\text{s.t. } Y_t = A_t K_t^{1-\alpha} L_t^\alpha \quad (3.A.2)$$

Firms labor and capital demand are given by:

$$L_t = \alpha \frac{Y_t}{w_t} \quad (3.A.3)$$

$$K_t = (1 - \alpha) \frac{Y_t}{r_t^k} \quad (3.A.4)$$

From where $Z_t = 0$, and output supply can be expressed by $Y_t = w_t L_t + r_t^k K_t$. The Objective function and budget constraints of household i are given by:

$$\max_{C_t^y(i), C_{t+1}^m(i), C_{t+2}^o(i), W_{t+2}^m(i)} \mathbb{E}_t \left\{ U(C_t^y(i)) + \beta U(C_{t+1}^m(i)) + \beta^2 \left[U(C_{t+2}^o(i)) + \frac{U(W_{t+2}^m(i))}{1 + \phi} \right] \right\} \quad (3.A.5)$$

$$\text{s.t. } C_t^y(i) = B_t^y(i) \quad (3.A.6)$$

$$C_{t+1}^m(i) = Y_{t+1}^m(i) - (1 + r_t) B_t^y(i) + B_{t+1}^m(i) - K_{t+1}(i) + \frac{Q_{t+1}^o(j)}{1 + g_t} \quad (3.A.7)$$

$$C_{t+2}^o(i) = -(1 + r_{t+1}) B_{t+1}^m(i) + (1 - \delta) K_{t+1}(i) - Q_{t+2}^o(i) \quad (3.A.8)$$

$$(1 + r_t) B_t^y(i) \leq D_t^y(i), \text{ where } D_t^y(i) = \theta Y_{t+1}^m(i)^\mu \quad (3.A.9)$$

$$Q_{t+1}^o(j) \geq 0, \text{ where } (j) \text{ represents household's (i) parents.} \quad (3.A.10)$$

where $U(C_t) = \frac{C_t^{1-\sigma}-1}{1-\sigma}$, and $Y_t^m(i) = w_t L_t^m(i) + r_t^k K_t(i)$. $W_{t+2}^m = Y_{t+2}^m + \frac{Q_{t+2}^o}{1+g_{t+2}}$, where $Y_{t+2}^m(i)$ is household i children expected income. From the first order conditions on W_{t+2}^m and C_{t+2}^o we derive an expression for bequests:

$$W_{t+2}^m = \frac{C_{t+2}^o}{(1+\phi)^{\frac{1}{\sigma}}} \Leftrightarrow \quad (3.A.11)$$

$$Q_{t+2}^o = (1+g_{t+1}) \left[\frac{C_{t+2}^o}{(1+\phi)^{\frac{1}{\sigma}}} - Y_{t+2}^m \right] \quad (3.A.12)$$

The consumption Euler equation is given by¹⁴:

$$\mathbb{E}_t \frac{C_{t+1}^o}{1+r_t} = \beta_{r_t} C_t^m \quad (3.A.13)$$

where,

$$\beta_{r_t} = \beta^{\frac{1}{\sigma}} (1+r_t)^{\frac{1-\sigma}{\sigma}} \stackrel{(\sigma=1)}{=} \beta \quad (3.A.14)$$

Using the production function and the first order conditions for K_{t+1} we derive the following expressions for w_t and r_t^k :

$$w_t = \alpha A_t^{\frac{1}{\alpha}} \left(\frac{1-\alpha}{r_t^k} \right)^{\frac{1-\alpha}{\alpha}} \quad (3.A.15)$$

$$r_t^k = \frac{r+\delta}{1+r} \quad (3.A.16)$$

In the case of the benchmark model we assume that children and parents labor endowment types are independent. Then for all household i we assume that, $Y_{t+2}^m(i) \equiv \bar{Y}_{t+2}^m = \frac{w_{t+2}}{\alpha} \bar{L}_{t+2}^m$. Using the expressions above we can derive the following expression for bequest if strictly positive, and equal to zero otherwise:

$$Q_{t+1}^o(i) = (1+r_t) \Psi_t \left[\alpha Y_t^m(i) - D_{t-1}(i) + \frac{Q_t^o(j)}{1+g_{t-1}} \right] - (1-\Psi_t)(1+g_t) \bar{Y}_{t+1}^m \quad (3.A.17)$$

¹⁴We can use this expression because the model is deterministic.

where the constant $\Psi_t = \frac{\beta_{r_t}}{\beta_{r_t} + \frac{(1+\phi)^{\frac{1}{\sigma}}(1+\beta_{r_t})}{1+g_{t+1}}} \in]0; 1[$ for $\phi \in]-1; +\infty[$.

We introduce income inequality in the model by assuming that there are two exogenous labor endowment types L^γ for the middle-aged, where $\gamma \in \{low, high\} \equiv \{l, h\}$.

Loan supply per middle age household has then the following expression:

$$L_t^s = \eta L_t^{s,l} + (1 - \eta) L_t^{s,h} \quad (3.A.18)$$

$$L_t^{s,\gamma} = -\bar{B}_t^{m,\gamma} \quad (3.A.19)$$

$$-B_t^{m,\gamma}(i) = \frac{\beta_{r_t}}{1 + \beta_{r_t}} \left[Y_t^{m,\gamma}(i) - D_{t-1}^{y,\gamma}(i) + \frac{Q_t^o(j)}{1 + g_{t-1}} - K_t^\gamma(i) \left(\frac{1 - r_t^k}{\beta_{r_t}} + 1 \right) \right] + \frac{Q_{t+1}^{o,\gamma}(i)}{(1 + \beta_{r_t})(1 + r_t)} \quad (3.A.20)$$

The marginal savings rates with an inoperative and operative bequest motive are given by:

$$\frac{\partial L_t^s}{\partial Y_t^m} = \frac{\beta_{r_t}}{1 + \beta_{r_t}} \left[\alpha - (1 - \alpha) \frac{1 - \delta}{\beta_{r_t}(1 + r_t)} \right] \quad (3.A.21)$$

$$\frac{\partial L_t^{s,Q}}{\partial Y_t^m} = \frac{\beta_{r_t}}{1 + \beta_{r_t}} \left[\alpha - (1 - \alpha) \frac{1 - \delta}{\beta_{r_t}(1 + r_t)} \right] + \frac{\alpha \Psi_t}{1 + \beta_{r_t}} \quad (3.A.22)$$

If all high types leave a bequest to their descendants, and all low types leave none loan supply aggregation in steady state is straight forward, and the difference of marginal savings rate between high and low types is given by:

$$\Delta MSR_t^{hl} = \frac{\alpha \Psi_t}{1 + \beta_{r_t}} > 0 \quad (3.A.23)$$

Although we do not enforce this mechanism we nevertheless present the conditions its verification. Given the bequest expression (3.A.17) it is possible to derive the value for the bequest received by one agent in period t when middle age, above which the bequest motive is operative for that agent during the next period:

$$Q_{min,t}^{m,\gamma} = \frac{1 + \beta_{r_t}}{\beta_{r_t}} \frac{(1 + \phi)^{\frac{1}{\sigma}}}{1 + r_t} \bar{Y}_{t+1}^m - (\alpha Y_t^{m,\gamma} - D_{t-1}^\gamma) \quad (3.A.24)$$

Received bequests by any household have a maximum value which is the steady state bequest of a high agent type where all its ascendants were of the high type too. This expression is given by:

$$Q_{max}^m = \frac{1}{\frac{1+g}{\Psi(1+r)} - 1} \left[(\alpha Y^h - D^h) - \frac{1 + \beta_r}{\beta_r} \frac{(1 + \phi)^{\frac{1}{\sigma}}}{1 + r} \bar{Y}^m \right] \quad (3.A.25)$$

In steady state, if $Q_{min,t}^{m,l} > Q_{max}^m$ then the bequest motive of all low income types is always inoperative, although they may receive some from their parents of high type. Moreover, if $Q_{min,t}^{m,h} < 0$ then all high type agents leave a bequest to their children independently of having received a bequest from their parents. Those two conditions are verified for any reasonable calibration of our model.

In this case the average steady state expressions for bequests of each type are given by:

$$\bar{Q}^{o,h} = \frac{1}{\frac{1}{\Psi(1+r)} - \frac{1-\eta}{1+g}} \left[(\alpha Y^h - D^h) - \frac{1 + \beta_r}{\beta_r} \frac{(1 + \phi)^{\frac{1}{\sigma}}}{1 + r} \bar{Y}^m \right] \quad (3.A.26)$$

$$Q^{o,l} = 0 \quad (3.A.27)$$

$$\bar{Q}^o = (1 - \eta) Q^{o,h} \quad (3.A.28)$$

3.B Intergenerations Utility of Consumption

We now consider the utility of children in the preference function. Utility is maximized subject to the same budget constraints as in previous sections, (3.3.2), (3.3.3), (3.3.4), (3.3.5) and (3.3.6), for households born at time t , and $t + 1$:

$$\max_{C_t^y, C_{t+1}^m, C_{t+2}^o, Q_{t+2}} \mathbb{E}_t \left\{ v_t + \frac{\beta}{1 + \phi} v_{t+1} \right\} \quad (3.B.1)$$

$$\text{where } v_t = \log(C_t^y) + \beta \log(C_{t+1}^m) + \beta^2 \log(C_{t+2}^o) \quad (3.B.2)$$

Assuming that the bequest zero lower bound is not binding, from FOC Q_{t+2} we derive the following expression relating middle age consumption of two consecutive generations:

$$\mathbb{E}_t \frac{C_{t+1}^m}{C_t^m} = \frac{\beta}{1+\phi} \frac{1+r_t}{1+g_{t+1}} = \frac{1+r_t}{1+r_t^b} \quad (3.B.3)$$

Where $1+r_t^b = (1+\phi)^{\frac{1+g_t}{\beta}}$. Then, in steady state, the equilibrium real interest rate with an operative bequest motive is given by:

$$1+r = 1+r^b = (1+\phi)^{\frac{1+g}{\beta}} \quad (3.B.4)$$

From loan market equilibrium we derive a general expression for the equilibrium real interest rate in steady state, which is strictly lower than r_n when the bequest motive is operative, and $Q > 0$:

$$1+r = \frac{1+\beta}{\beta} \frac{(1+g)D}{(Y^m - D) + \left(\frac{1}{1+g} + \frac{1}{\beta(1+r)} \right) Q} = 1+r^b < 1+r^n \quad (3.B.5)$$

In that case the bequest in steady state is given by:

$$Q = \left(\frac{\beta}{2+\phi} \right) (r^n - r^b)(Y^m - D) \quad (3.B.6)$$

Consequently, in steady state, the bequest motive is operative if and only if the no-bequest natural rate of interest $r^n = \frac{1+\beta}{\beta} \frac{(1+g)D}{(Y^m - D)}$ is greater than $r^b = (1+\phi)^{\frac{1+g}{\beta}}$. Otherwise, when $r^n \leq r^b \Rightarrow Q = 0$ and $r = r^n$. Then, if the steady state no-bequest natural rate of interest r^n as in Eggertsson and Mehrotra [21] decreases below the threshold r^b the bequest motive of households become inoperative.

Inequality and low interest rates

We now consider two household income types h and l as in previous sections, and assume that average income is constant, so that $Y^m = \eta_s Y_t^{m,l} + (1 - \eta_s) Y_t^{m,h}$. We

further assume that parents born at time $t - 1$ receive no bequests at time t , and children don't leave bequests to future generations at time $t + 2$. From the budget constraints we derive the following expressions for middle age consumption at time t and $t + 1$ for parent's and children of types i and j respectively:

$$(1 + \beta)C_t^{m,i} = Y_t^{m,i} - D_{t-1} - \frac{Q_{t+1}^i}{1 + r_t} \quad (3.B.7)$$

$$(1 + \beta)C_{t+1}^{m,j} = Y_{t+1}^{m,j} - D_t + \frac{Q_{t+1}^i}{1 + g_{t+1}} \quad (3.B.8)$$

Combining equation (3.B.3) with expressions (3.B.7), (3.B.8), Q_{t+1}^i is given by:

$$Q_{t+1}^i = \gamma [(1 + r_t)(Y_t^{m,i} - D_{t-1}) - (1 + r_t^b)(Y_{t+1}^{m,j} - D_t)] \quad (3.B.9)$$

where $\gamma = \frac{1}{1 + \frac{1+\phi}{\beta}}$. Loan supply for household type i when the bequest motive is operative has the following expression:

$$L_t^{s,i} = \frac{\beta}{1 + \beta} \left[(Y_t^{m,i} - D_{t-1}) \left(1 + \frac{\gamma}{\beta} \right) - \frac{\gamma}{\beta} \frac{1 + r_t^b}{1 + r_t} (Y_{t+1}^{m,j} - D_t) \right] \quad (3.B.10)$$

From where we can directly state that loan supply of an household expands if the descendants are expected to become poorer, or the agent becomes richer. The bequest motive is operative, $Q_{t+1}^i > 0$, if:

$$(1 + r_t^b) \frac{Y_{t+1}^{m,j} - D_t}{Y_t^{m,i} - D_{t-1}} = 1 + r_t^{ij} < 1 + r_t \quad (3.B.11)$$

Note that $r_t^{ij} \leq r_t^{lh}$, so that when $r_t > r_t^{lh}$ bequest motives of all agents in this economy are operative. As long as this relation persists, aggregate loan supply and equilibrium real interest rate are not affected by a change in inequality as we can observe from expressions below:

$$L_t^s = \frac{\beta}{1 + \beta} \left[(Y^m - D_{t-1}) \left(1 + \frac{\gamma}{\beta} \right) - \frac{\gamma}{\beta} \frac{1 + r_t^b}{1 + r_t} (Y^m - D_t) \right] \quad (3.B.12)$$

The corresponding equilibrium real interest is given by:

$$1 + r_t = \frac{1 + \beta}{\beta} \frac{(1 + g_t)D_t + \frac{\gamma(1+r_t^b)}{1+\beta}(Y^m - D_t)}{(Y^m - D_{t-1}) \left(1 + \frac{\gamma}{\beta}\right)} = 1 + r_t^* \quad (3.B.13)$$

Then,

$$\forall \{i; j\}, Q_t^i > 0 \text{ if } \frac{Y_{t+1}^{m,j} - D_t}{Y_t^{m,i} - D_{t-1}} < \frac{1 + r_t^*}{1 + r_t^b} \quad (3.B.14)$$

If children inherit their parents type, $Y_{t+1}^{m,j} = Y_t^{m,i}$, and we assume $D_t = D_{t-1} = D$ then $r_t^{ll} = r_t^{hh} = r_t^b$, the zero lower bound bequest threshold is the same for both types, and the equilibrium real interest rate $r_t = \min(r_t^n, r_t^*)$, is always unaffected by changes in income inequality.

Otherwise, if children and parents types are *iid*, and parents cannot predict their children type then the problem from the parent perspective is the same as the one previously specified, using an *average* income type for children with an endowment equal to the constant weighed average population endowment Y^m . Bequests are positive for both types if

$$\forall i Q_t^i > 0 \text{ if } \frac{Y^m - D_t}{Y_t^{m,i} - D_{t-1}} < \frac{1 + r_t^*}{1 + r_t^b} \Leftrightarrow \quad (3.B.15)$$

$$Y_t^{m,i} > Y_t^* = \frac{1 + r_t^b}{1 + r_t^*} (Y^m - D_t) + D_{t-1} \quad (3.B.16)$$

An increase of income inequality not affecting Y^m does not affect aggregate loan supply, demand, and the equilibrium real interest rate, as long as all endowments remain above the threshold Y_t^* .

If income inequality increases at time t so that the bequest zero lower bound for low income type becomes binding, or $Y_t^{m,l} < Y_t^* \Rightarrow Q^l = 0$ and $Y_t^{m,h} > Y_t^*$, then loan

supply expressions for each household type are given by:

$$L_t^{s,l} = \frac{\beta}{1+\beta} (Y_t^{m,l} - D_{t-1}) \quad (3.B.17)$$

$$L_t^{s,h} = \frac{\beta}{1+\beta} \left[(Y_t^{m,h} - D_{t-1}) \left(1 + \frac{\gamma}{\beta} \right) - \frac{\gamma}{\beta} \frac{1+r_t^b}{1+r_t} (Y^m - D_t) \right] \quad (3.B.18)$$

Aggregate Loan supply becomes a positive function of the high type income, and expands with inequality increases:

$$L_t^s = \frac{\beta}{1+\beta} (Y^m - D_{t-1}) + (1-\eta_s) \frac{\gamma}{1+\beta} \left[(\textcolor{red}{Y}_t^{m,h} - D_{t-1}) - \frac{1+r_t^b}{1+r_t} (Y^m - D_t) \right] \quad (3.B.19)$$

The natural rate of interest, derived from equilibrium in the loan market, becomes a decreasing function of the high type income:

$$1+r_t = \frac{1+\beta}{\beta} \frac{(1+g_t)D_t + (1-\eta_s) \frac{\gamma(1+r_t^b)}{1+\beta} (Y^m - D_t)}{(Y^m - D_{t-1}) + (1-\eta_s) \frac{\gamma}{\beta} (\textcolor{red}{Y}_t^{m,h} - D_{t-1})} < 1+r_t^* \quad (3.B.20)$$

When inequality increases above a certain level the bequest zero lower bound of poor becomes binding, and their marginal savings with respect to income decreases relative to the rich. Then the net effect on aggregate loan supply of an increase of income inequality will be positive, and negative on the equilibrium real interest rate. The mechanism is the same as the one presented in previous sections, but with a different preference function.

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